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APPLICATION OF DIFFERENTIAL GAMES TO PROBLEMS

OF MILITARY CONFLICT:

TACTICAL ALLOCATION PROBLEMS - PART II

by

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ABSTRACT

The mathematical theory of optimal control/differential games is used to study the structure of optimal allocation policies for some tactical allocation problems with combat described by Lanchester-type equations of warfare. Both deterministic and stochastic attrition processes are considered. For the optimal control of deterministic Lanchester-type attrition processes, a general solution algorithm for the synthesis of the optimal policy is developed. Optimal allocation policies are developed for numerous one-sided optimization problems of tactical interest in order to study the dependence of the structure of these optimal policies on model form. Consideration has been given to singular extremals, multiple extremals (including dispersal surfaces), and state variable inequality constraints. It is shown how to apply the theory of state variable inequality constraints to determine the optimal control of deterministic Lanchester-type processes in order to treat non-negativity restrictions on force levels and thus to study the dependence of optimal policies upon the force levels. Various attrition models are considered (reflecting different assumptions as to target acquisition process, command and control capabilities, target engagement process, variations in range capabilities of weapon systems). Solutions are developed for Lanchester-type equations of modern warfare with variable attrition-rate coefficients. The optimal control of the Lanchester stochastic process is studied.

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Prepared by:

PREFACE

This report documents my research on tactical allocation in dynamic combat situations over the past two years under the sponsorship of the Office of Naval Research (both as part of the Foundation Research Program at the Naval Postgraduate School and under contracts NR 276-027 and PO 2-0150). Some of this work is recent; some of it is not so recent. In all cases, however, the documentation is new and does not duplicate any past reports.

Much of this work has been submitted for publication in the open literature. However, because of long "turn around times" for refereeing in several journals (sometimes over a year before receiving the referees' initial comments), I felt that it was appropriate to include several manuscripts as appendices that had been previously submitted for publication in the open literature. Hence, some of this work is not of recent vintage: the paper "On the Solution to Lanchester-Type Equations With Variable Coefficients" was submitted for publication to <u>Operations Research</u> in January 1971. In many instances I have obtained far more extensive results but have not had the time to fully document them.

Thus, the appendices of this report are more or less separate entities in themselves and may be read independently of each other. Accordingly, each appendix contains its own list of references and reference numbers apply only within the appendix in which they appear.

A similar remark applies to the numbering of equations. Also, the pagination is separate for each individual appendix.

The emphasis in this report is on the applications of control theory to tactical allocation problems in dynamic combat situations. Although consideration of the various problems contained in this report has required knowledge of research on the frontiers of deterministic optimal control theory, I have felt it to be inappropriate for the scope of this research to supply proofs for any optimization theory results. However, I have tried to give the appropriate references to support all statements of theory contained herein.

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1. Introduction.

This report documents research performed under the sponsorship of the Office of Naval Research (contract PO 2-0150) during the time period of 27 March 1972 to 16 June 1972. We discuss the results of a study of some idealized models for the optimization of combat dynamics and their possible implications for defense planners. Our research approach has been to combine Lanchester-type formulations of combat attrition (both deterministic and stochastic) and generalized control theory [64] (both deterministic and stochastic optimal control, dynamic programming, differential games).

A quantitative theory of tactical allocation is developed through the examination of a sequence of simplified models. These combat models are too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent computer simulation studies or field experimentation. The effects of various modelling assumptions upon the structure of the optimal allocation policies are systematically studied by contrasting the solutions for various models.

A major result of our research is that optimal tactical allocation policies are quite sensitive to the precise type of model adopted, even as to whether the tactical scenario lasts a specified period of time or terminates only when a pre-determined state has been reached. Insights are provided into such important questions as:

(1) How should targets be selected?

(2) Do target priorities change with time?

(3) Do battle termination circumstances affect the optimal allocation policies?

(4) How does the nature of the attrition process affect target selection?

(5) What is the effect of ammunition constraints?

(6) How does the uncertainty and confusion of combat affect the optimal selection rules?

To develop our theory of tectical allocation we have extensively studied some specific combat scenarios. Optimal tactics for the following have been studied: selection of target type at which to fire, regulation of firing rate. The influences of the following factors have all been considered:

(1) combatant objectives (form of criterion function and valuation of surviving forces),

(2) weapon system performance characteristics,

(3) termination conditions of conflict,

(4) force strengths,

(5) type of attrition process,

(6) effect of resource constraints,

(7) range capabilities of weapon systems.

The tactical situations are described by deterministic Lanchester-type equations of warfare. The combat continues over a period of time with a choice of tactics available to both sides and subject to change over time. The mathematical theory of deterministic optimal control/

differential games has been used to solve the problems under consideration. Thus, it seems appropriate to discuss these techniques briefly.

a. Differential Games/Optimal Control.

The theories of optimal control and differential games were developed for optimization problems in which the (deterministic) system's dynamics are described by a system of ordinary differential equations. The reader can find references additional to those given here and a brief review of past developments in our previous report [113].

In a two-person zero-sum deterministic differential game (henceforth abbreviated to simply differential game) each player chooses strategies (for precise definitions, see [12] or [52]) in order to maximize his own criterion functional (which when added to that of his opponent yields zero) for a system whose dynamics is governed by a system of ordinary differential equations. An example problem relevant to our research on tactical allocation problems is

$$\underset{\psi_{ij}(t)}{\text{maximize}}\ \underset{\phi_{ij}(t)}{\text{minimize}}\left\{\sum_{i=1}^{n} w_i y_i(T) - \sum_{i=1}^{m} v_i x_i(T)\right\} \quad \text{with } T \text{ specified,}$$

$$\text{subject to:}\quad \frac{dx_i}{dt} = r_i - \sum_{j=1}^{n} \psi_{ij} a_{ij} y_j \quad \text{for } i = 1,\dots,m,$$

$$\frac{dy_i}{dt} = s_i - \sum_{j=1}^{m} \phi_{ij} b_{ij} x_j \quad \text{for } i = 1,\dots,n,$$

with

$$\sum_{i=1}^{m} \psi_{ij} = 1 \quad \text{for} \quad j = 1,\ldots,n,$$

$$\sum_{i=1}^{n} \phi_{ij} = 1 \quad \text{for} \quad j = 1,\ldots,m,$$

$$\phi_{ij},\psi_{ij} \geq 0 \quad \text{and} \quad x_i,y_i \geq 0, \tag{1}$$

where

$x_i(t), y_i(t)$ are force levels,

r_i, s_i are replacement rates,

v_i, w_i are the utilities assigned survivors,

a_{ij} is the rate at which one Y_j unit can destroy X_i,

b_{ij} is the rate at which one X_j unit can destroy Y_i,

ϕ_{ij} is the fraction of X_j who fire at Y_i,

and ψ_{ij} is the fraction of Y_j who fire at X_i.

For our illustrative purposes here, let us assume that the battle is scheduled to last a prescribed duration of time (we further assume that no side is annihilated before T). Such problems (mainly pursuit and evasion) have been extensively studied in recent years. The goal of this research has been to study tactical allocation problems using such models.

Pioneering research on differential games was done by R. Isaacs and culminated in his book [72]. Isaacs developed a general principle (the tenet of transition) in order to develop his "main equation"

(which is basically a two-sided extension of the Hamilton-Jacobi equation (see [52])). L. Berkovitz [12] has justified in a rigorous manner many of Isaacs' original results, which were heuristically obtained. In [10] Berkovitz developed many basic results for differential games according to variational arguments.

The above theories apply to differential games which possess solutions in pure strategies. It is a simple matter to devise problems for which this is not true (see [11], [13], [52]). However, Isaacs conjectured that when the Hamiltonian is separable, i.e. a function independent of ϕ plus a function independent of ψ, then the differential game has a solution in pure strategies.* This conjecture was later proven to be true by L. Berkovitz (see pp. 170-173 of [10]) and A. Friedman [52]. We shall see below that the Hamiltonian for (1) will be separable in this sense.

Let us consider the above example (1) again. The Hamiltonian is given by

$$H(t,x,p,\phi,\psi) = \sum_{i=1}^{m} p_i\{r_i - \sum_{j=1}^{n} \psi_{ij}a_{ij}y_j\}$$

$$+ \sum_{i=1}^{n} q_i\{s_i - \sum_{j=1}^{m} \phi_{ij}b_{ij}x_j\}, \tag{2}$$

where

$p_i(t)$ denotes the dual variable corresponding to x_i,

and $q_i(t)$ denotes the dual variable corresponding to y_i.

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*We use the symbol ϕ to denote all the variables under control of X.

In the case when the differential game has a saddle-point (in pure strategy solutions) so that a value exists, let us denote the value function as $W(t,x,y)$. This value function is well-known [12] to be C^1 (i.e. continuously differentiable with respect to each of its arguments) except at certain manifolds of discontinuity. Except at such manifolds, one can make the identification

$$p_i^*(t) = \frac{\partial W}{\partial x_i}(t),$$

and

$$q_i^*(t) = \frac{\partial W}{\partial y_i}(t).$$

From (2), we see Hamiltonian is indeed separable in the sense discussed above. However, as not noted by Isaacs, Berkovitz, or Friedman, the above expression (2) for the Hamiltonian function is appropriate only when none of the state variable inequality constraints (SVIC's) of (1) is active.* In Appendix E, we discuss the fact that although problems with SVIC's have been extensively studied in the literature of deterministic optimal control theory (our study of tactical allocation problems in the Lanchester theory of combat, however, has led to the uncovering of an important gap in the existing theory of SVIC's [115]), a current gap in the theory of differential games is the lack of treatment of problems (such as (1) above) with inequality constraints on functions of the state variables only (the control variables not

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*In (1) x_i for $i = 1,\ldots,m$ and y_j for $j = 1,\ldots,n$ are state variables, while the ϕ's and ψ's are called control variables. An example of a SVIC is the condition that $x_1 \geq 0$, i.e. the X_1 force level must be non-negative. Thus, (2) is appropriate only for $x_i > 0$ and $y_i > 0$ for all i.

explicitly appearing in these inequality constraints). In fact, the approach that R. Isaacs used to solve the "War of Attrition and Attack" (see pp. 96-104 in [72]) is easily shown to be inadequate for other similar problems (for further details see Appendix E). (This important gap in existing theory and the inadequacy of Isaacs' (and others) treatment of SVIC's is not noted in the recent work of S. Sternberg [111].)

Thus, when $x_i > 0$ and $y_i > 0$ for all i, the Hamiltonian is given by (2). Hence the Hamiltonian is separable (in the sense discussed above) so that a solution in pure strategies can be found to (1). Let us pursue further details of solution development to illustrate additional difficulties.

Assuming that $x_i > 0$ and $y_i > 0$ for all i, we determine an extremal strategy pair, denoted as (ϕ^*,ψ^*), by the max-min principle

$$\underset{\psi_{ij}}{\text{Maximum}}\ \underset{\phi_{ij}}{\text{minimum}}\ H(t,x,p^*,\phi,\psi) = H(t,x,p^*,\phi^*,\psi^*). \tag{3}$$

This leads to the problem

$$\text{maximize}\left\{\sum_{i=1}^{m} \psi_{ij} a_{ij}(-p_i^*)\right\}$$

$$\text{subject to:}\quad \sum_{i=1}^{m} \psi_{ij} = 1$$

$$\psi_{ij} \ge 0 \quad \text{for } j = 1,\ldots,n, \tag{4}$$

which may be routinely solved to yield

$$\psi^*_{ij}(t) = \delta_{i,k(j,t)}, \tag{5}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and $k(j,t)$ is the index such that

$$a_{kj}(-p^*_k) = \text{maximum}\{a_{1j}(-p^*_1), \ldots, a_{mj}(-p^*_m)\}, \quad \text{for } j = 1, \ldots, n. \tag{6}$$

Similarly we obtain

$$\text{minimize}\{\sum_{i=1}^{n} \phi_{ij} b_{ij} q^*_i\}$$

$$\text{subject to:} \quad \sum_{i=1}^{n} \phi_{ij} = 1$$

$$\phi_{ij} \geq 0 \quad \text{for } j = 1, \ldots, m, \tag{7}$$

so that $\phi^*_{ij}(t)$ is analogously determined.

It may be that the index $k(j,t)$ is not unique, i.e. the linear program (4) has alternate optima. This causes no difficulty unless this situation continues for a finite interval of time. When this happens, we say that the corresponding segment of the battle trajectory is a <u>singular subarc</u>. Of great practical significance is the fact that the max-min principle does not provide adequate tests

for whether a singular subarc can yield an optimal outcome. Thus, the possibility of singular subarcs must be investigated in tactical allocation differential games, since the control (decision) variables appear linearly in problems such as (1). We shall discuss this important aspect further below (see Appendix D for further details in one-sided case), but for now let us simply state that it may be shown that it is not possible to have a singular solution to (1).

From (5) we see that this model (1) yields that the optimal tactic is to concentrate all fire on one enemy target type. However, we really don't gain that much insight into changes in the optimal target selection policy with time, since we don't easily see explicitly how the dual variables (and hence the optimal tactic via (4) and (7)) change over time. The adjoint system of differential equations for the dual variables is, of course, given by

$$\frac{dp_i^*}{dt} = -\frac{\partial H}{\partial x_i}(t,x,p^*,\phi^*,\psi^*) = \sum_{k=1}^{n} \phi_{ki}^* b_{ki} q_k^* \quad \text{for} \quad i = 1,\ldots,m$$

$$\frac{dq_j^*}{dt} = -\frac{\partial H}{\partial y_j}(t,x,p^*,\phi^*,\psi^*) = \sum_{k=1}^{m} \psi_{kj}^* a_{kj} p_k^* \quad \text{for} \quad j = 1,\ldots,n \tag{8}$$

with

$$p_i^*(t=T) = -v_i$$

$$q_j^*(t=T) = w_j.$$

Hence, at the end of battle at $t = T$ we have

$$\psi^*_{ij}(t=T) = \delta_{i,k(j,T)}, \tag{9}$$

where $k(j,T)$ is the index such that

$$a_{kj}v_k = \text{maximum}\{a_{1j}v_1,\ldots,a_{mj}v_m\}, \quad \text{for} \quad j = 1,\ldots,n,$$

and

$$\phi^*_{ij}(t=T) = \delta_{i,\ell(j,T)}, \tag{10}$$

where $\ell(j,T)$ is the index such that

$$b_{\ell j}w_\ell = \text{maximum}\{b_{1j}w_1,\ldots,b_{nj}w_n\}, \quad \text{for} \quad j = 1,\ldots,m.$$

In Appendix C, we show how the extremal (candidate for optimal control) control may be explicitly determined as a function of time for a one-sided analogue of the problem at hand.

Thus, we see that for separable controls as occur in target selection problems in the Lanchester theory of combat the solution method (i.e. the max-min principle) essentially (speaking somewhat imprecisely now) reduces the problem of determining the solution of the differential game to solving two "separated" one-sided optimization (optimal control or variational) problems. Moreover, in [10] L. Berkovitz reduces a differential game to two such control problems (problems

of Bolza with differential inequalities as side conditions) in order to deduce necessary conditions for optimal paths.*

It should therefore be clear that if there are difficulties (either theoretical or computational) in solving such optimal control problems, then there will be difficulties in solving the corresponding differential game. (Alternatively, we may think of an optimal control problem as a differential game in which the strategy(s) of one of the two players is fixed beforehand.) Accordingly, our research approach has been to study optimal target selection problems for dynamic combat situations with decisions available to both sides by first considering corresponding one-sided (optimal control) problems. After the difficult points have been mastered in the simple one-sided problems, then the differential games are most profitably tackled.

In the next section we discuss some aspects of deterministic optimal control problems that can cause difficulties. All these subtle points may also arise in differential games. We have found that by studying both the theory and examples of difficult aspects in one-sided problems we have appreciably increased our ability to solve corresponding differential games. In fact, as noted above, one of our research findings is that a theory of state variable inequality constraints (which is essential in problems such as (1) because of the

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*This is done by the famous method of Valentine (see [125]). M. Hestenes [61] formulated the (now standard) general control problem and deduced necessary conditions (including the maximum principle (frequently referred to as the Pontryagin maximum principle)) according to this program in 1949. The same approach is readily extended to nonlinear programming (including linear programming), but this apparently has been overlooked by workers in the field. However, our recent paper [118] remedies this gap.

requirement that force levels be non-negative) is lacking in the current theories for differential games.* We plan to develop such a theory in the future.

Based on this intimate relationship between the mathematical theories of differential games and optimal control [63], our research approach (as stated above) has been to consider one-sided versions of some tactical allocation structures before tackling the more realistic two-sided decision problem. Our intent has been to make sure that both theoretical and computational aspects are firmly established for these optimal control problems before attempting to solve the more complex differential game versions of such problems.

Optimal control theory has emerged as a vigorous discipline since World War II. Professor Magnus Hestenes of the USA apparently first gave the now standard control formulation for a variational problem (as well as developed optimality conditions) and urged others to follow his approach in 1949 [61]. Professor Hestenes did not feel that it was appropriate to publish this work in the open literature, since, for example, his development of a "Weierstrass maximum principle" was a translation of well-known results from the classical calculus of variations. Consequently, this pioneering work [61] received little

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*A. Friedman (see Chapter 6 of [52]) has developed a non-variational theory of differential games via a limit approach. Although he has established that a certain class of differential games (our problem (1) belongs to this class) with state variable inequality constraints (SVIC's) do possess a value and the optimal strategies (see pp. 238-240 of [52]) does not appear to be adequate. Specifically, Friedman makes an unsupported assumption about a property of optimal strategies (the assumption that it is optimal to have $x_1(T) > 0$).

attention from researchers other than some at RAND. In 1956 Pontryagin and his associates announced their celebrated "maximum principle" [19]. This event ushered in over a decade of intense activity on variational methods. Due to this intense activity in the field of optimal control theory during the past 15 years and our specific interest in applications to military operations research problems, it does not seem appropriate to cite here voluminous research articles (however, see [3] and [99] for surveys). The current state of the art in deterministic optimal control theory is reflected in the following contemporary textbooks and monographs from which a further more specialized guide to the literature can be obtained [4], [6], [18], [28], [40], [42], [45], [51], [62], [89], [91], [101].

b. Special Features of Optimal Control Problem.

As we have discussed above, our research approach has been to initially study dynamic tactical allocation problems by considering one-sided optimal control versions of them. Moreover, these deterministic optimal control problems that we have studied within the framework of the Lanchester theory of combat have contained certain special features that have caused difficulty (both theoretically and computationally). Let us consider some examples to illustrate such difficulties.

For illustrative purposes, we first consider the following optimal control problem for target selection in Lanchester combat (see [117] (reproduced in this report as Appendix A) for further discussion of model)

$$\underset{\phi(t)}{\text{maximize}}\{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ unspecified,}$$

subject to: $$\frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2 \tag{11}$$

$0 \le \phi \le 1$ (inequality constraint on control variable),

$x_1, x_2, y \ge 0$ (state variable inequality constraints),

where

$x_1(t)$, $x_2(t)$, $y(t)$ are force levels,

p, q, r are utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

and ϕ is the fraction of Y fire directed at X_1.

The stopping rule for the battle is that the conflict terminates at $t = T$ defined by

(a) $y(T) = 0$,

or (b) $x_1(T) = x_2(T) = 0$.

Further analysis yields that there are five cases which must be considered:

C_1: $x_1(T) = 0$, $x_2(T) > 0$, $y(T) = 0$,

C_2: $x_1(T) = 0$ before $x_2(T) = 0$, $y(T) > 0$,

C_3: $x_1(T) = 0$ after $x_2(T) = 0$, $y(T) > 0$,

C_4: $x_1(T) > 0$, $x_2(T) = 0$, $y(T) = 0$,

C_5: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$.

A constraint such as $x_1 \geq 0$ is referred to as a state variable inequality constraint (SVIC from [29]). When such a constraint is active (as when $x_1(t) = 0$ for $t_1 \leq t \leq t_2$), the usual necessary conditions of optimality (maximum principle) require modification: the adjoint equations are modified, boundary conditions for the dual variables require special treatment, juncture (corner) conditions must be satisfied, some special multiplier conditions must be satisfied (see Appendix E for details). This area of deterministic optimal control theory (problems with SVIC's) has had much recent research activity (see bibliographies in [79], [115]). However, the appropriate treatment of a SVIC in order to determine the optimal trajectory is apparently not widely known among practitioners of optimal control (as pointed out by our recent note [115]), although the appropriate necessary conditions (as well as sufficient conditions) are, of course, well-known by active researchers.

When $x_1, x_2, y > 0$, the Hamiltonian for the problem (11) is given by

$$H(t,x,p,\phi) = -p_1(t)\{\phi a_1 y\} - p_2(t)\{(1-\phi)a_2 y\} - p_3(t)\{b_1x_1 + b_2x_2\}, \quad (12)$$

where $p_1(t)$ is the dual variable corresponding to x_1, etc. By (12) we see that the Hamiltonian is a linear function of the control variable so that a singular solution [80], [81] is possible. Singular solutions to target selection problems in the Lanchester theory of combat are discussed in Appendix D. This aspect of the optimal control of deterministic Lanchester attrition processes is of great practical significance, since it means that it is possible to have other than $\phi^* = 0$ or 1 as an optimal fire distribution policy (in contradiction to the results presented in [111]).

Let us further elaborate upon singular solutions in deterministic problems of optimal control. In such a problem, the maximum principle may fail to determine an optimal trajectory, since the maximization of the Hamiltonian may not lead to a well-defined expression for optimal control [80], [81] (also see Chapter 8 of [28]). Singular solutions usually occur when the Hamiltonian is a linear function of the control variables (as is the case for fire distribution problems in the Lanchester theory of combat as formulated by Isbell and Marlow [74] and Weiss (see pp. 94-95 of [129])). However, all problems for which the Hamiltonian is a linear function of the control variables do not have singular subarcs (see below) in their solution. In particular, we shall see that problem (11) does not.

The above problem (11) has one control variable (denoted as ϕ), and it appears linearly in the Hamiltonian. By a <u>singular subarc</u> we

denote that part of an optimal trajectory on which the maximum principle cannot be used to determine the control because the coefficient of the control variable in the Hamiltonian is zero (see pp. 226-227 of [80]). Then the term "singular solution" will be used to refer to any optimal trajectory which contains one or more singular subarcs.

To elaborate further, when the Hamiltonian H is a linear function of the control variable ϕ, then if $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time (or, another way to say this, the coefficient of ϕ vanishes identically for a finite interval of time), then the maximum principle does not determine the control. When this situation occurs, neither the maximum principle nor the classical variational theory provide adequate tests for maximality of the singular subarc. However, Kelly [82], [83] generalized the Legendre-Clebsch condition of local optimality of extremal paths containing singular subarcs (see [75], [78], [110] for very recent developments).

Observe that on a singular subarc for which $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time <u>all</u> feasible values of ϕ maximize the Hamiltonian. On a singular subarc we determine the singular control by requiring that we remain on the singular subarc, i.e. $\frac{\partial H}{\partial \phi}$ remains zero. If $\frac{\partial H}{\partial \phi}$ is to be identically equal to zero for a finite interval of time, then all of its derivatives with respect to time must also be equal to zero. We determine the singular control, which keeps the system on the singular subarc, by considering as many of the

time derivatives of $\frac{\partial H}{\partial \phi}$ as are required for the control variable ϕ to appear explicitly so that it may be determined from an algebraic equation. Thus, in general we consider

$$0 = \frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = \ldots \tag{13}$$

We must further check to see that we can get a maximum return (in the case when we wish to maximize the criterion functional) from use of the candidate singular subarc. The following condition (generalized Legendre-Clebsch condition) is necessary for a singular subarc to yield a maximum return

$$(-1)^k \frac{\partial}{\partial \phi} \left\{\frac{d^{2k}}{dt^{2k}}\left(\frac{\partial H}{\partial \phi}\right)\right\} \leq 0. \tag{14}$$

It is obtained by examining the negative semidefiniteness of the second variation for a special class of explicitly defined control variations [83]. For the problem (15) considered below, it suffices to consider the generalized Legendre-Clebsch condition with $k = 1$.

For fire distribution problems in the Lanchester theory of combat it turns out that when the rate of target-type attrition is proportional to the number of enemy firers only (as in the above problem (11)), it is impossible (excluding the pathological case when $a_1 b_1 = a_2 b_2$) to have $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$. Thus, it is impossible to have a singular solution. (However, such important aspects were not discussed in [111]*).

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*This is a common gap in applications of optimal control theory by many practitioners. For example, the vital subject of singular solutions is not discussed in [1], [2], [67], or [106], even though singular solutions are treated by heuristic means.

Our research has yielded the fact that the possibility of singular solutions must be investigated for all such fire distribution problems in the Lanchester theory of combat. In particular, a singular solution arises in the following problem

$$\max_{\phi(t)} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 x_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2 \tag{15}$$

$$0 \le \phi \le 1 \quad \text{and} \quad x_1, x_2, y \ge 0.$$

In Appendix D we show that it is possible to have an optimal fire distribution policy that is not an extreme point in the control variable space, e.g. $\phi^* = a_2/(a_1+a_2)$. For the problem at hand, this explicit expression for the singular control is obtained from the equation

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = 0,$$

by use of the conditions $\frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ and the canonical equations (i.e. both state and adjoint system).

A more subtle difficulty in obtaining a complete solution to fire distribution problems is that there may be more than one extremal

(an extremal is a path on which the necessary conditions of optimality (maximum principle) are almost everywhere satisfied) leading from an initial point in the force level (state) space to the terminal surface. This occurs both in the Isbell-Marlow problem [74], [117] and the supporting weapon system game of H. K. Weiss [130], [122]. Such a situation occurs when it is possible to reach different points on the terminal surface from the same point in the initial state space (see pp. 276-284 of [4]). Such extremals are difficult to identify, since sufficient conditions of optimality [53] may be satisfied along each of the multiple extremals.

In problem (11) above, there are circumstances under which more than one extremal (battle trajectory on which the necessary conditions are satisfied) leads from the same initial force levels x_1^o, x_2^o, y_0 to the terminal surface. For example, under the appropriate circumstances extremals may lead to both C_1 and C_5.* Moreover, it may be shown that sufficient conditions of optimality [53] are satisfied for each extremal. However, this difficulty in solution development is probably best treated by example so that we defer further in depth discussion until the appendices.

To summarize, we have identified three aspects of fire distribution problems in the Lanchester theory of combat that have not been adequately treated previously. These aspects are as follows:

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*Actually, there are two types of extremals that may lead to C_5.

(1) state variable inequality constraints,

(2) singular subarcs,

(3) multiple extremals.

Moreover, each of these aspects can be vital in developing solutions in applications.

2. Objectives of Research.

The general objective of this research is to determine the structure of the optimal fire distribution/target selection policies for various specific scenarios of tactical interest. The specific objective of this research on applications of control theory to Lanchester-type formulations of combat attrition is to determine the sensitivity of target selection and force allocation strategies to the form of the attrition model and to the time history of force levels.

3. Related Work of Others.

Additionally, we will review in more depth developments in the quantitative analysis of military tactics/strategies through Lanchester-type models of warfare (i.e. variational models to determine optimal allocations (e.g. target selection, fire distribution, etc.) as a function of time and/or force levels in combat between heterogeneous forces). Hence, we consider the related work of others in the fields of

(a) Lanchester-type models of warfare,

(b) differential games, and

(c) optimal control of Lanchester attrition processes.

We are aware of probably well over 200 titles of papers and books which contain information that, in one sense or another, would probably assist in the attainment of the above stated objectives of this research. For the present we shall highlight some major works. In this survey we consider the subject area of differential games to be representative of applicable material within the more extensive framework of "generalized control theory" (see [64], where this term was coined by Y. C. Ho). More extensive reviews of pertinent literature are to be found in two of our past reports [112], [113].

a. Lanchester-Type Models of Warfare.

One of the earliest attempts to establish a mathematical model of the dynamics of mass combat was by Lanchester [87] in 1916. He postulated several deterministic models that were a system of ordinary differential equations which related the strengths of opposing military forces to length of combat. During World War II B. Koopman extended Lanchester's results and also suggested a reformulation of the problem in stochastic form [96]. After World War II the RAND Corporation carried on further studies [60], [97], [98] whose results were summarized by R. Snow [109]. H. K. Weiss (then at Aberdeen Proving Ground) and others [5], [26], [46], [58], [59], [128], [129] have subsequently extended deterministic Lanchester models.

R. Brown developed models for the stochastic analysis of combat [27]. The relationship between the above mentioned stochastic and deterministic Lanchester formulations was pointed out relatively early in their development (see [109], for example) but is probably best discussed in a recent paper by B. O. Koopman [86]. The first probability analysis of Lanchester-type equations apparently appeared in the (now classical) Morse and Kimball book [96]. Using a stochastic model, an expression was obtained for the probability of one side winning in a fight to the finish in which the random attrition process corresponds to a deterministic Lanchester linear-law attrition process. Stochastic and deterministic model results are compared. A more in depth comparison of stochastic and deterministic model results has been given by G. Weiss [127]. R. Brown developed not only expressions for the probability of winning but also approximations to the probability of winning as determined by stochastic models of combat attrition corresponding to both the Lanchester linear law and the square law [27]. Further extensions of these results have been given by Smith [108] and Kisi and Hirose [85]. To date, the most thorough comparison of deterministic and stochastic homogeneous-force Lanchester formulations has been by G. Clark (see Chapter 11 of [16] and [41]), who has also developed the time-state solution for the stochastic attrition process corresponding to a deterministic Lanchester linear-law process [41].

S. Bonder [22] did the pioneering work on the determining the Lanchester attrition-rate coefficient (for weapon systems that adjust fire based on the results of the immediately preceding round). C. Barfoot [7] later suggested defining the Lanchester attrition-rate coefficient as the harmonic mean of the attrition rates (or, equivalently, the reciprocal of the expected time to kill a single target). Barfoot also presented an alternate and more general method for computing the Lanchester attrition-rate coefficient. Bonder [23] then showed that the mean attrition rate for Markov-fire weapons is readily obtainable by the methods described in [22] when one defines the Lanchester attrition-rate coefficient as the reciprocal of the expected time to kill a single target. Kimbleton [85] has extended the above work by deriving the distribution of the time required to destroy a target. Other papers [104], [103] have discussed the estimation of parameters for the Lanchester attrition-rate coefficient.

A good review of the Lanchester theory of combat is by Dolansky [46], and this includes a comprehensive list of references through 1965. A recent thesis directed by us [56] contains a fairly comprehensive bibliography of articles appearing in the open literature since 1964. Recent trends in the Lanchester theory of combat were discussed at a 1967 NATO Conference [48], [126]. Of particular relevance to our recent work is the pioneering work of D. Etter [49] (also to appear in [48]), who has formulated and (partially) solved several allocation of fire problems (see below for further details). Etter

has also given consideration to the spatial dimension of combat and force mobility [49].

H. K. Weiss [129] extended Lanchester-type equations to include the relative movement of two homogeneous forces, allowing time and space to be "traded" for casualties. He considered the two attrition-rate coefficients to be dependent upon force separation in such a way that their ratio was constant. S. Bonder [20], [21] used Weiss' extension to study the effects of mobility and various range dependencies for the attrition-rate coefficients on the number of surviving forces. He has studied scenarios for combat between two homogeneous forces, and developed solutions for the case of a constant attack velocity under special circumstances. In our past research, we have shown [114] that a simple closed-form solution is possible in all those cases when the ratio of attrition-rate coefficients is constant, regardless of whether the attack velocity is constant or not (see also [116]). We have subsequently developed [123] a completely general solution to variable coefficient Lanchester-type equations for combat between two homogeneous forces and further developed analytic results in the following important cases: (1) opposing weapon systems whose kill rates have different range dependencies and (2) weapons systems with different effective ranges. Some related work appears in [24] and [43].

b. <u>Differential Games</u>.

The study of differential games was initiated by R. Isaacs at RAND in the early 1950's [68], [69], [70], [71], but his work has not

been available to a wide audience until more recently [72]. His basic concept, "the tenet of transition," is generalization of Bellman's [8] "Principle of Optimality" to a competitive environment, and this is used to develop necessary conditions for optimal strategies. A more recent and more rigorous development of these basic necessary conditions is by Berkovitz [12].

Since the excellent paper by Ho, Bryson, and Baron [65] in 1965, there has been a literal explosion of papers on differential games but almost all deal exclusively with pursuit-evasion problems. Excellent survey papers which bear this out are by Simakova (Russian literature) [107] and Berkovitz [13]. (However, applications to the Lanchester theory of combat are discussed in the next section.) Besides Isaacs' book, others which treat differential games are by Blaquiere et al [17] (extension of their geometric approach to optimal control), Bryson and Ho [28] (Chapter 9), and Friedman [52]. This latter work appears to be the most authoritative one on the mathematical theory of differential games to date and contains a fairly comprehensive bibliography for the purely mathematical aspects of differential games.

c. Optimal Control of Lanchester Attrition Processes.

In 1964 Dolansky [46] noted that an underdeveloped area in the Lanchester theory of combat was optimal target selection in combat between heterogeneous forces (optimal control/differential games). Our past results [117], [122] have extended all the previously published

results [74], [130] in the open literature cited by Dolansky. Recently, we have extended these results even further [119], [120], [121].

In [117] we showed that the solution originally obtained by Isbell and Marlow [74] (also given in [111]) is incorrect for a certain range of model parameters and presented a general methodology for solving all such problems. In [122] we extended this methodology to differential games by considering the supporting weapon system game of H. K. Weiss [130], which was previously only treated by heuristic means. (It was not noted in [111] that Weiss did not apply the mathematical theory of differential games to obtain his solution to this problem.)

More recently, we have discussed the influence of various factors (combatant objectives, termination conditions of conflict, type of attrition process, variable attrition-rate coefficients, and limited ammunition) [119]. In [120] we presented some preliminary results for optimal fire distribution/target selection in combat between heterogeneous forces (the special case of a homogeneous force against heterogeneous enemy forces). We considered both constant attrition-rate coefficients and some special cases of variable attrition-rate coefficients. In [121] we treated a simple problem of target selection for a homogeneous force in Lanchester combat against two enemy force types each of which undergoes attrition at a rate proportional to the product of the numbers of firers and targets.

Some similar problems had been previously studied by D. Etter [49] (also to appear in [48]). Although not using modern optimal control theory, his arguments (which failed to give consideration to the long-run battle outcome) did lead to results which in many cases anticipated many of our recent results. Etter [49] pointed out the fundamental difference between the optimal allocation of fire with square-law attrition and with linear-law attrition and has done some computational studies.

Work in this area has also been done at the University of Michigan [24]. However, the dependence of optimal strategies upon model form (i.e. type of attrition process, battle termination conditions, replacements, combatant objectives, range dependent attrition rates, etc.) was not noted (our results are reported in Appendix C of [113] and [117]) and certain subtleties of optimal target selection policies not detected. Moreover, in [111] S. Sternberg apparently gave the same solution to a two-on-one problem as originally obtained by Isbell and Marlow [74] and presented the same solution development methodology as they did. In our past research [117] (reproduced for the reader's convenience in this report as Appendix A (see also Appendix F)) we have shown that the solution originally obtained by Isbell and Marlow [74] is not completely correct. Hence, Sternberg's claim (see p. 153 of [111]) that a "two-on-one example was then solved in its entirety" does not seem justified to us.

In view of the similarities in the research areas considered by Sternberg [111] and the research results presented in report, it seems appropriate to make a few critical comments about Sternberg's work. As we have noted above, Sternberg did not develop solution methodology capable of obtaining a complete solution to the two-versus-one fire distribution problem. Even more serious, however, is his failure to note the well-known fact [63], [65] that differential games are a class of two-sided optimal control problems. It is not surprising then that Sternberg does not discuss the well-known control theory results for singular solutions, state variable inequality constraints (SVIC's), non-uniqueness of extremals, etc., as applied to differential games in the Lanchester theory of combat. Furthermore Sternberg [111] does not note that a theory for state variable inequality constraints has not been developed for differential games or that Weiss [130] did not use the necessary conditions of optimality to develop his "solution" to the supporting weapon system game.

The practical significance of the above facts is that, for example, the solution of the war of attrition and attack given by Isaacs in [72] (and partially reproduced by Sternberg on pp. 17-25 of [111]) is not justified (the treatment of the boundary conditions for the dual variables is inadequate as is that for constrained subarcs when a SVIC is active). (It is easy to give an example (see Appendix E) for which Isaacs' approach does not yield the correct solution.) Hence, Sternberg's attempt to solve a "two-on-two" differential game

appears premature to us.* Our own research indicates that use of the appropriate necessary conditions of optimality appreciably reduces the volume of allocation strategy combinations that have to be considered.

Additionally, Sternberg [111] did not investigate the dependence of optimal allocation strategies upon model form. His major research conclusion (on pp. 153-154 of [111]) that optimal allocation strategies in Lanchester-type processes are always 0, 1 is incorrect as stated, since it is easily shown that for two-versus-one combat in which target types undergo attrition at rates proportional to the product of the numbers of firers and targets, the optimal allocation strategy may be other than 0 or 1 (see [121]). Thus, Sternberg's conclusion must be qualified be stating that his results apply to, for example, Lanchester-type processes in which the rate of attrition of each target type is proportional to (the sum over all firer classes of) the number of firers only (speaking somewhat imprecisely, Lanchester square-law processes). It was not noted by Sternberg that the structure of the optimal allocation policy may be different in prescribed duration battles and fights to the finish.

Finally, Bonder and Farrell (see pp. 27-28 of [24]) state that (for Lanchester attrition processes in which target type attrition rate due to each class of firers is proportional to the number of firers

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*In Appendix E of this report we develop a theory of SVIC's for one-sided fire distribution problems in the Lanchester theory of combat. Some of the results we give are not to be found in the literature. As we have done in the past [115], we hope to publish some of these results in the control theory literature.

only) the optimal target selection policy is independent of the number of weapons in the firer or target group. Our research results (see [117] or Section 9 of Appendix G) contradict the above statement made by Bonder and Farrell: the optimal fire distribution policy may depend <u>indirectly</u> upon force levels.

Other similar tactical/strategic allocation problems that have been studied using differential game models are the supporting weapon system game of H. K. Weiss [130], missile warfare (counter-value versus counter-force targeting) [105], and the logistics allocation game of Moglewer and Payne [95]. We have already noted that Weiss did not use the subsequently well-known necessary conditions of optimality to develop the solution in his pioneering work [130]. Our previous work [122] (see also Appendix B of [113]) corrects this gap. Several authors [38], [39], [66] (see also pp. 54-57 of [105]) have considered differential equation models of a strategic missile exchange and have attempted to study optimal tactics via differential game theory. Moglewer and Payne [95] consider a similar model for the optimal allocation of logistics resources. However, there are many gaps in the above work. For example, Moglewer and Payne [95] consider (in our terminology presented in [117]) a terminal control differential game. However, they fail to determine the domains of controllability for terminal states and express optimal tactics in terms of <u>terminal force levels</u>. In [117] we showed that it is essential to express optimal tactics in terms of <u>initial force levels</u>.

It seems appropriate to briefly touch upon the implications for operational gaming (see [100] or [124] for discussion of terminology and background) of our analytic study of tactical allocation problems. We believe that the primary value of operational gaming is in formulating the problem. (Accordingly, we are coordinating our analytic efforts with the operational study work of P. Chaiken of Stanford Research Institute (Naval Warfare Research Center).) Moreover, we agree with L. Berkovitz and M. Dresher in their opinion (see p. 612 of [14]) that "operational gaming is not a helpful device for solving a game or getting significant information about the solution." We believe that insights into optimal allocation strategies are to be obtained through analytic work such as ours.

Finally, we could find no previous work on the optimal control of the Lanchester stochastic process (i.e. stochastic versions of the problems discussed above).

4. <u>Overview of Research Program</u>.

In this section we briefly discuss the various subject areas that we have considered in developing our quantitative theory of tactical allocation.

a. <u>Combat Attrition Models</u>.

In our research on tactical allocation we have given consideration to the various different types of models that have been proposed (hypothesized) in the literature to describe combat between two homoheneous forces. Such homogeneous force formulations are readily extended to describe the

attrition for fire distribution problems in combat between heterogeneous forces. Our purpose in considering the various modelling alternatives for combat attrition has been to determine the sensitivity of optimal target selection and force allocation strategies to the form of the attrition model. Based upon our research, we conclude that optimal allocation policies are significantly affected by the nature of the attrition model.

The reader should recall that there are two general approaches which may be taken to the modelling of the attrition process:

(a) deterministic formulation which takes the form of a system of first order ordinary differential equations, and

(b) stochastic formulation which views the casualty process as a Markov process.

In the remainder of this section we shall discuss deterministic formulations as related to our study of allocation strategies. Stochastic combat attrition models are discussed in the next section.

Besides our review of the literature above, the interested reader can find comprehensive reviews in [41], [46], [56]. The various forms that the attrition rates can take in models of combat between two homogeneous forces has been discussed by Lanchester [88], Snow [109], H. Weiss [129], Brackney [26], Willard [131], and Helmbold [59]. (The reader can find a further discussion in Appendix H.) In our past research [117], [120], [121] we have studied optimal target selection policies when the attrition rate of each target type is proportional to:

(a) the number of firers only,

(b) the product of the numbers of firers and targets.

These correspond to the classical Lanchester square-law process and linear-law process, respectively.

R. Helmbold [59] has proposed a general formulation of combat attrition for combat between two homogeneous forces (based upon his empirical study of a large number of land battles [57]). His general model [59] accounts for inefficiencies of scale when force sizes are grossly unequal. In this report we present some preliminary results for optimal target selection when Helmbold's general model is used to describe attrition in combat between heterogeneous forces. In doing this work we have extended his formulation to combat between heterogeneous forces and present some analytic solutions.

b. Deterministic Versus Stochastic Models of Combat Attrition.

As indicated by our review of the pertinent literature and noted above, two basic approaches to analytic war gaming are (1) the Lanchester-type deterministic differential equation models* [24], [25], [50] or (2) stochastic models (such as DYNTACS [16], a high resolution simulation). Almost all previously published research on tactical allocation problems** (including our own work) has treated attrition as a deterministic process. We have briefly discussed work on stochastic formulations in the review of pertinent literature above. In Section 2

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*These are frequently referred to as expected-value models.

**The one exception of a stochastic optimization problem is the work of Isbell and Marlow [73].

of Appendix I the reader can find a more comprehensive review of past work on the Lanchester stochastic process. There are, however, conflicting views in the literature as to whether these two approaches yield the same results. (We certainly don't plan to try to settle this difficult question, but wish to caution the reader that there is not universal agreement among workers in the field about this important question.)

G. Clark has studied (see Chapter 11 of [16] and [41]) the difference between deterministic and stochastic homogeneous-force Lanchester formulations and concludes that (see p. 11-19 of [16]) "the deterministic model would have difficulty approximating a stochastic simulation" with respect to the time history of force levels. Based upon our study of Clark's work [41], we raise the question as to whether there may be significant differences between optimal allocation policies for deterministic and stochastic attrition processes. If such a difference were to be observed, then this would raise the more basic question of whether or not different study results could be obtained merely by the choice of modelling technique, i.e. deterministic or stochastic formulations.

c. Special Features of Deterministic Optimal Control Models for Tactical Allocation.

In Section 1. above, we have discussed three important problem areas in applying deterministic optimal control theory to study tactical allocation problems

(a) non-uniqueness of extremals,

(b) state variable inequality constraints,

(c) singular subarcs.

(These topics were not discussed in [111].) Most of our research effort has been directed at developing a satisfactory treatment of these topics for tactical allocation problems. All are relatively near the frontiers of research in deterministic optimal control theory.

Further detailed discussions of the above topics can be found in the appropriate appendices which are as follows:

(a) non-uniqueness of extremals: Appendices A, F, G,

(b) state variable inequality constraints: Appendix E,

(c) singular subarcs: Appendix D.

It seems appropriate, however, to give further general discussion of these aspects in order to emphasize their importance in the optimization of combat dynamics via optimal control theory. Additionally, we will review their current status within deterministic optimal control theory.

(1) <u>Non-uniqueness of Extremals</u>.

The purpose of this research is to determine optimal fire distribution/target selection policies for tactical allocation problems. However, as is well-known, the maximum principle (or the max-min principle for two-sided (differential game) problems) provides only necessary conditions of optimality. Moreover, let the reader recall that we refer to a path on which the necessary conditions of optimality

(maximum printiple) are almost everywhere satisfied as an extremal. Thus, in order to determine the optimal allocation policy one must show that a particular extremal is indeed an optimal trajectory (otherwise one may incorrectly identify a non-optimal policy as being optimal).

Two ways of determining optimal trajectories are as follows:*

(a) check whether sufficient conditions of optimality [92], [53] are satisfied on the extremal,

(b)** by citing the appropriate existence theorem, show that an optimal control exists to the problem at hand; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering a finite number of alternatives.

It turns out, however, that the former approach is inadequate for most tactical allocation problems that we have considered, since existing sufficient conditions [92], [53] only apply to optimal control problems of fixed length (i.e. fixed terminal time). Without going into subtle details at this time, let us state that the results of [92], [53] cannot be applied to, for example, the Isbell-Marlow problem [74], [117] without qualification (if they are, it is easy to falsely conclude that certain non-optimal extremals are optimal).

The second approach given above is the one we have taken in this research. If an extremal is unique, then it is optimal and no difficulty

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*For the purposes of our expository discussion here we are speaking somewhat imprecisely in order to communicate to the general reader why certain topics have received so much attention in our research.

**This is essentially a condensation of the general solution procedure given in Appendix A (and further extended in Appendix G).

exists. However, in the simplest fire distribution problem we have shown that extremals may not be unique [117]. (Apparently, this was not noted in any past research on this problem [74], [111].) Moreover, we have shown how to determine the optimal allocation policy when there are multiple extremals in such a problem (see Appendices A and F for further details). In the latter case, one may have to contend with a dispersal surface (see pp. 132-141 of [72]) being present in the solution.

In our past research, we have shown that dispersal surfaces may exist in the solution to even the simplest fire distribution problem [117]. This rare singular surface is very difficult to determine, since its presence in a solution cannot be determined directly by the maximum principle but requires "considerations in the large." Besides the examples (all two-sided problems) given in Isaacs' book (see pp. 135-155 of [72]), we are aware of only one other problem [44] studied in the literature in whose solution a dispersal surface has been found.

The existence of multiple (i.e. non-uniqueness of) extremals in tactical allocation problems within the Lanchester theory of combat has important implications for computational methods (such as dynamic programming or other finite difference approximation techniques). Approximate solutions to optimal control problems may be developed by discretizing time (i.e. using finite difference approximations) and treating the resulting problem as a mathematical programming problem [30], [93]. The difficulties caused by the existence of multiple extremals

or dispersal surfaces in the solution to a continuous-time optimal control problem are not circumvented by considering an approximating mathematical programming (finite-dimensional optimization) problem.

With such subtle features being present in even the simplest fire distribution problem, the reader might well ask, "Can anything be done to circumvent such mathematical complexities?" We believe that this can be partially done by identifying the circumstances under which such difficult mathematical points arise in the model's solution. In other words, it is very important to have complete (and mathematically correct) solutions to the simplest tactical allocation problems if for no other reason than to identify which formulations lead to prohibitive mathematical complexities (and hence should be avoided). For example, we know that for n-versus-one combat such difficulties don't arise when surviving target types are valued in direct proportion to their kill capability as measured by their Lanchester attrition-rate coefficient.

Finally, except for the references cited above (and pp. 276-284 of [4]), we could find no other treatment of multiple extremals in the current literature of deterministic optimal control theory.

(2) <u>State Variable Inequality Constraints</u>.

As we saw in our discussions in Section 1 above, all tactical allocation problems contain negativity restrictions on the force levels due to physical reasons (negative force levels don't make sense). In general, force levels will be represented by <u>state variables</u> so that

such a restriction is mathematically called a state variable inequality constraint (frequently denoted as SVIC). The maximum principle (in its original form [101]) is inadequate to handle SVIC's and special mathematical theories have been developed for state-variable-inequality-constrained problems. When we started our research, we found that results were widely scattered in the literature and that a completely adequate theory to solve even the simplest fire distribution problem did not exist. Hence, we have developed a theory of SVIC's for tactical allocation problems within the Lanchester theory of combat (see Appendix E).

A mathematical theory of SVIC's is essential for solving tactical allocation problems within the Lanchester theory of combat. Using such a theory, one can determine when it is not optimal to drive a force level to zero, the optimal order in which to annihilate target types, etc. Thus, the mathematical theory (necessary conditions) allows one to (significantly) reduce the volume of allocation strategy combinations which one has to examine in solving a differential game (thus circumventing the computational difficulty encountered by Sternberg (see pp. 154-155 of [111])).

As noted frequently in this report, no adequate theory of SVIC's exists within the current mathematical theory of differential games.*

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*A. Friedman, of course, has treated differential games with SVIC's (i.e. games with restricted phase coordinates)[52]. His approach is different than the variational approach used in optimal control theory, and his emphasis is on existence theorems. Moreover, his treatment (see pp. 239-240 of [52]) of Isaacs' war of attrition and attack (see pp. 96-104 of [72]) implicitly requires an unsupported assumption.

The development of a mathematical theory of SVIC's for differential games which occur within the Lanchester theory of combat appears to be the most needed extension of the existing theory of differential games for military operations research applications.

Although optimal control problems involving inequality constraints on a function of the state variables with no explicit dependence upon the control variables have received fairly comprehensive treatment in recent years (see [77], [79], [94], [115] for comprehensive bibliographies), most results are widely scattered in the literature. (Moreover, the theory which has appeared in textbooks has some important gaps in it and is inadequate (by itself) to solve the fire distribution problems that have been the subject of our research [115].) The poineering work of the Russian R. Gamkrelidze in 1959 (see Chapter VI in [101]) was followed in the USA by that of L. Berkovitz [9], S. Dreyfus [47], and Bryson, Denham, and Dreyfus [29]. Berkovitz and Dreyfus [15] subsequently have shown the equivalence of the results of Gamkrelidze and Berkovitz with those of Dreyfus.

McIntyre and Paiewonsky [94] have written an excellent survey article on the theory of SVIC's and summarize theoretical results through 1966. Subsequent work has been by Jacobson and Lele [77] and Jacobson, Lele, and Speyer [79], the latter authors deriving their results by considering generalized Kuhn-Tucker conditions in a Banach space. Sufficient conditions of optimality for problems with SVIC's

have been given by Funk and Gilbert [53]. Finally, we would like to mention the work by two students of M. Hestenes [55], [102].

Moreover, as we have recently pointed out [115], practitioners of optimal control have not always been aware of the precise content of the above work. We are aware of no previous work relating the theory of SVIC's to allocation problems in the Lanchester theory of combat.

(3) Singular Subarcs.

As we have seen in Section 1 above, tactical allocation problems within the Lanchester theory of combat may be formulated (see equations (1) above) as differential games in which the strategy variables (e.g. fraction of X_j who fire at Y_i) appear linearly. Such problems are called singular (because the matrix of second partial derivatives of the Hamiltonian with respect to the strategy variables is a singular matrix) and require special treatment. One-sided versions of such fire distribution problems then are singluar problems of optimal control. A discussion of terminology and the mathematical aspects of singular optimal control problems has been given in Section 1.b. above. We saw there that the maximum principle does not determine the singular control and that the classical theory does not provide adequate tests of optimality for singular subarcs.

A mathematical theory of singular subarcs is essential for solving tactical allocation problems within the Lanchester theory of combat because it is required to determine the optimality of fire

distribution policies that are not extreme points of the control variable space (i.e. policies that are other than 0 or 1). We have seen that an optimal fire distribution policy, denoted as ϕ^*, such that $0 < \phi^* < 1$ arises in the simple problem of target selection for a homogeneous force in Lanchester combat against two enemy force types each of which undergoes attrition at a rate proportional to the product of the numbers of firers and targets [120].

The establishment of the singular subarcs in such a problem as above requires examination of the generalized Legendre-Clebsch condition (a second order condition). In a problem such as (11) in which target-type attrition corresponds to Lanchester's classical square-law attrition process (i.e. the attrition rate of a target type is proportional only to the number of firers) the absence of singular subarcs is established by showing that it is impossible to have (except for pathological cases) $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$, where H denotes the Hamiltonian. The importance of the generalized Legendre-Clebsch condition is emphasized by the fact that we have encountered a problem (suggested by the work of P. Chaiken) in which it is the only way to establish the non-optimality of splitting forces in combat.

Singular problems of optimal control were apparently first explicitly studied in 1963 by Johnson and Gibson [81] (see [80] for a discussion of singular solutions in optimal control problems within a broad perspective of applied mathematics). Since that time the subject has grown to comparative maturity although most recent results

are widely scattered in the literature. Surveys of early work are contained in the works by Kelley, Kopp, and Moyer [83] (summarizes results through 1966) and Bryson and Ho (see Chapter 8 of [28]) (summarizes results through 1968). The above work is sufficient to adequately treat all the singular problems of the optimal control of deterministic Lanchester processes which we have so far encountered.

Kelly [82], [83] apparently first generalized the Legendre-Clebsch necessary conditions to singular subarcs. This was further generalized to singular extremals involving several control variables by Goh [54]. Jacobson later derived an additional necessary condition via differential dynamic programming [75]. Very recently, Jacobson [76] has given a general sufficiency theorem for the second variation which allows one to treat both singular and non-singular problems. This yields conditions analogous to the well-known no-conjugate-point condition (see pp. 181-184 of [28]) for singular subarcs and has led to necessary and sufficient conditions of optimality for singular control problems [110] (see also [78]).

Except for our own work [121], we are aware of no previous work applying the theory of singular extremals to tactical allocation problems within the Lanchester theory of combat.

d. Optimal Control of the Lanchester Stochastic Process.

A totally underdeveloped area in the Lanchester theory of combat has been optimal target section in combat among heterogeneous forces when attrition is a stochastic process. Based on our review of the

pertinent literature (see Section 1 of Appendix I), we conclude that there has been no past adequate treatment of stochastic versions of the fire distribution problems that we have previously studied using deterministic models. By studying such stochastic versions, we plan to see if the "best" tactics for target selection are dependent upon how attrition is modelled, i.e. deterministic or stochastic process. In this report we present preliminary results (development of the fundamental functional equation for the optimal expected value function, its analytic solution in some special cases, and a numerical approximation for the general case). Although our results are special cases of well-known results in the field of optimal stochastic control, the optimal control of the Lanchester stochastic process had apparently never been studied in the past by this approach.

In Appendix I we consider a battle which is to last a prescribed length of time as an optimal stochastic control problem. Our idea is to compare the structure of the optimal allocation policies between deterministic and stochastic formulations. Accordingly, we have developed a complete "solution" to the corresponding deterministic control problem (see Appendix G). We are currently directing a graduate student (LT R. Powers, USN) in making a numerical comparison between results obtained by these two models, and we will report these results at a later date.

5. Guided Tour of the Appendices.

In this section we summarize the work which is contained in the appendices and explain why this work was done. In our first report [113] we emphasized the extension of results which had previously appeared in the literature. In the present report we give further extensions and, moreover, develop the mathematical foundations for more extensive future work.

In Appendix A we derive a complete solution to the Isbell and Marlow fire programming problem [74]. This current work revises earlier preliminary results on this problem. We develop general methodology for solving such deterministic optimal control problems and then demonstrate its application by developing a complete solution (however, further extensions are given in Appendix F) for a terminal control battle (the battle only terminates when the course of battle has reached some specified state) between a homogeneous force and a heterogeneous enemy force of two target types. Our methodology is more general, though, and may be applied to the cases of one-versus-n target types, variable attrition-rate coefficients, etc.

The structure of the optimal fire distribution policy for the Isbell-Marlow problem is discussed. We show how the optimal fire distribution policy may depend indirectly upon the force levels. For the attrition model used (attrition rate of target types proportional to the number of firers only) the optimal policy is always to concentrate all fire on one target type. Our work extends that of Isbell

and Marlow [74] by showing that the solution to this optimal control problem contains dispersal furfaces. Additionally, we show how to determine the domain of controllability for extremals to each terminal state.

The work presented in Appendix A was done because the Isbell-Marlow problem [74] represents the simplest terminal control fire distribution problem. Any mathematical theory developed to handle more complex tactical allocation problems of greater practical significance must be able to adequately handle this simple case.

In Appendix B we examine the structure of optimal allocation policies for tactical situations described by Lanchester-type models of warfare by studying a sequence of simplified models. We do this because no previous work had ever systematically investigated the dependence of tactics upon model form. We consider models for two types of choice situations

(1) selection of target type,

(2) regulation of firing rate.

These problems are solved by the mathematical theory of optimal control.

In the first sequence of models we examine the effects on the optimal target selection policy of the following factors: objectives of the combatants, termination conditions of the conflict, number of target types, some special cases of time-dependent attrition-rate coefficients, and type of attrition process. We then examine a sequence

of models to see how ammunition limitations affect firing rates. Next we discuss two-sided extensions of such problems but point out the value of studying one-sided problems as considered in this report. Finally, various implications of the models are discussed. Some of the results presented in this appendix are further extended in Appendix G.

In Appendix C we develop partial solutions to a sequence of models for the optimal fire distribution policy for a homogeneous force in combat against geterogeneous enemy forces. We consider a prescribed duration battle and concentrate on the special case when no force type is annihilated during combat. For both one-versus-two-target-types combat and one-versus-n-target-types combat we develop our results for both constant attrition-rate coefficients and also some special cases of variable attrition-coefficients.

We undertook the work of Appendix C to show how our results for one-versus-two-target-types combat could be extended to one-versus-n combat and variable attrition-rate coefficients (thus closer modelling real world complexities). In this work we draw heavily upon our research on homogeneous force models with variable attrition-rate coefficients [114], [116], [123] (see also Appendix H). First, we consider the problem of optimal fire distribution over two target types for some special cases of variable attrition-rate coefficients. Next, we consider several target types and constant attrition-rate coefficients. Then, we treat the more general case of several target types for some special instances of variable

attrition-rate coefficients. Finally, we make some observations on the results presented in this appendix.

Our results presented in Appendix C show that some special cases of variable attrition-rate coefficients are mathematically tractable for optimal control problems within the Lanchester theory of combat. Additionally, we show that in one-versus-n combat the solution to the optimal fire distribution problem takes a particularly simple form when enemy target types are valued (linearly) in direct proportion to their kill capability (as measured by the Lanchester attrition-rate coefficient). In this special case, the optimal fire distribution policy is particularly simple: namely, concentrate all fire upon the available target type with the largest product of attrition-rate coefficients (i.e. largest $a_i b_i$). This is even true for some special cases of variable attrition-rate coefficients.

In Appendix D we treat a simple problem of fire distribution for a homogeneous force in Lanchester combat against two enemy force types, each of which undergoes attrition at a rate proportional to the product of the numbers of firers and targets. We undertook this work in order to study the dependence of optimal tactics upon the attrition model. In addition to the Pontryagin maximum principle, the theory of singular extremals is required to solve this problem.

Our major result is that for fire distribution problems in which target-type attrition corresponds to the classical Lanchester linear-law attrition model "the optimal fire distribution policy is

not always to concentrate all fire on a single target type. There are circumstances when the optimal policy is to split one's fire between the two target types. Additionally, we show how to synthesize the optimal fire distribution policy from the basic optimality conditions. For constant attrition-rate coefficients we show that whether or not changes can occur in target priorities depends solely on how survivors are valued and is independent of the type of attrition process.

In Appendix E we summarize results for the theory of state variable inequality constraints (frequently denoted as SVIC's) which had previously been widely scattered in the literature. Then we apply this theory to two tactical allocation problems. The first problem (tactical air-war campaign) is a one-sided version of R. Isaacs' "War of Attrition and Attack" [72]. However, the non-negativity of the force levels was not adequately handled mathematically in past work [72], and we consider versions of the problem for which the previous method of analysis does not lead to an optimal policy. The second problem is the fire programming problem studied by Isbell and Marlow [74]. We consider several versions of both these problems. For one-versus-n-target-types combat we extend our past results by developing necessary conditions of optimality for annihilating a target type in two important cases: (1) two target types with a special case of variable attrition-rate coefficients and (2) n target types with constant attrition-rate coefficients.

We undertook the work presented in Appendix E since no past work on optimal tactical allocation had given an adequate treatment of the non-negativity restrictions on force levels in developing optimal policies. In our own previous work on the Isbell-Marlow problem [74] we followed the heuristic treatment originated by Isbell and Marlow [74] (who, in turn, were following Isaacs [70]) for determining the boundary conditions for the dual variables. Our work here establishes a firm theoretical foundation for such important aspects and justifies certain assumptions that we made in developing our solution to the Isbell-Marlow problem [117]. Thus, we re-consider the development of certain key necessary conditions of optimality within the framework of the theory of SVIC's.

In Appendix F we consider some further aspects of the solution to the Isbell and Marlow fire programming problem. We show how the Isbell-Marlow problem is solved using the theory of SVIC's. Additionally, we show that for certain values of model parameters (1) the optimal policy may not be unique and (2) in treating multiple extremals there is a dominated return (i.e. the return associated with one family of extremals dominates that associated with another family of extremals) besides the dispersal surfaces in this problem's solution. We undertook this work because our treatment of more complex tactical allocation problems relies heavily on the complete solution of simpler problems. This important theory (SVIC's) had never been applied previously to determine the optimal control of deterministic Lanchester processes.

In Appendix G we develop a complete solution to the prescribed duration battle using the theory of SVIC's. We do this in order to compare the structure of the optimal fire distribution policy for the prescribed duration battle problem with that of other problems (i.e. terminal control battle and optimal control of the Lanchester stochastic process). We give a general solution algorithm that applies to all such deterministic optimal control problems. We have used this algorithm to generate all the results given in this report. Discussions of the following important topics are given : (1) the structure of the optimal fire distribution policy and (2) the development of solutions to such problems. We show that even when target types undergo a "square-law" attrition process, the optimal allocation policy may depend indirectly upon the force levels.

In Appendix H we develop solutions to extensions of F. W. Lanchester's classical equations of modern warfare (frequently referred to as aimed-fire equations) for combat between two homogeneous forces. In these extensions the lethality of the fire (as expressed by the Lanchester attrition-rate coefficient) depends upon time. When the dependence is arbitrary, the solution is an infinite series of recursively related integrals; in special cases, more convenient representations (including representation in terms of tabulated functions) are available. Solutions are obtained in the following cases: (1) lethality of each side's fire proportional to a power of time and both lethalities initially zero, and (2) lethality of each side's fire

linear with time but only one side's lethality initially zero. The latter case models the constant speed approach between forces whose weapons have different maximum effective ranges. We undertook this work because we could find no adequate treatment available in the literature. These results have been subsequently used for problems on the optimal control of deterministic Lanchester processes with variable attrition-rate coefficients.

In Appendix I we present preliminary results on the optimal control of the Lanchester stochastic process. We consider a stochastic version of the prescribed duration fire distribution problem studied in Appendix G. Our purpose for doing this work is to determine whether the structure of the optimal allocation policy is affected by whether the attrition process is modelled as being deterministic or stochastic.

Thus, in Appendix I we develop the fundamental functional equation satisfied by the optimal expected value function and derive solutions in some special cases. Except for these special cases, the fundamental functional equation (actually a system of differential-difference equations) possesses a solution that is too complex for the development of a closed-form analytic solution. Therefore we have developed a finite difference approximation which can be used to generate numerical solutions for small numbers of combatants.

Using the above finite difference approximation, a thesis student has done some numerical computations for us. It has only proven to

be feasible to use the computer to generate numerical results for battles with up to 10 Y's versus 10 X_1's and 10 X_2's due to computer memory requirements. Additionally, we have considered approximations valid for large numbers of combatants. In pursuing this approach, we have developed a diffusion approximation to the Lanchester stochastic process.

In the fire distribution problems studied above for a homogenous force in combat with a heterogeneous enemy it has always been assumed that fire can always be instantaneously re-distributed from being entirely concentrated on one target type to any other one. For example, in "two-on-one" combat the decision (or control) variable represents the fraction of Y fire that is directed against X_1. Instantaneous jumps in this control variable (for example, from 0 to 1) have been permitted. This corresponds, in a sense, to a "perfect" command and control capability. In Appendix J we present preliminary results for the more realistic assumption that there is a limit to how fast ϕ can be changed corresponding to command and control limitations.

We undertook this work presented in Appendix J in order to explore the effects of command and control capabilities upon the structure of the optimal fire distribution policies. We show that the presence of command and control limitations (which cause the rate of re-distribution of fire to be bounded) causes fire to be shifted earlier than when fires can be instantaneously re-distributed in

anticipation of changes in target priorities. This phenomenon is analogous to the situation when with reduced reaction times, one tries to "anticipate" the actions of an enemy. The development of these results has required some advanced theory for SVIC's (treatment of a second order SVIC).

Finally, in Appendix K we further investigate the sensitivity of the optimal target selection policy to the form of the combat attrition model by considering a two-against-one-target-type selection problem with combat attrition following a general form proposed by R. Helmbold. For combat between homogeneous forces Helmbold [59] has proposed a general Lanchester-type model in which the effectiveness of a force is dependent upon the force ratio. This general formulation yields Lanchester's square law and Lanchester's linear law as special cases. We extend Helmbold's model to combat between heterogeneous forces and use this extension as the attrition model in a tactical allocation optimization problem.

In Appendix K, then, we present preliminary results for optimal target selection for Helmbold's general attrition structure. These preliminary results indicate that when target-type attrition follows Helmbold's general model, the <u>optimal policy is always to concentrate all fire on a single target type</u>. (In other words, we could find no singular solution in this case.) This should be contrasted with our result [121] that when target types undergo attrition corresponding to a "linear-law" Lanchester process, it is sometimes optimal to split

one's fire. In other words, as far as the optimal distribution of fire is concerned, Helmbold's general attrition structure behaves like that in which a target type's attrition rate is proportional only to the number of firers.

6. Relevance to Current Navy Problems.

In this section we explain the relevance of our research on the optimal control of Lanchester processes to current Navy problems.

a. Our theoretical work on tactical allocation problems is being interfaced with the ONR supported study effort of P. Chaiken of Stanford Research Institute. Chaiken has pioneered [31], [32], [33], [34] in the application of quantitative (Lanchester-type) methodology for force level planning, development of contingency plans, etc. This methodology has been used for significant force level problems defined by the military--Commander-In-Chief-Pacific (CINPAC), Headquarters Marine Corps (HQMC), and Chief of Naval Operations (CNO) [34].

To determine optimal allocation strategies in dynamic problems, Chaiken uses a heuristic optimization approach of assuming a certain type of strategy (constant or piecewise constant) and then comparing computed campaign outcomes for various levels of strategy variables (essentially, a so-called direct method of optimization [90]). Our research supports his study objectives by (1) justifying such assumptions, (2) suggesting strategy alternatives to explore in more detailed computer simulations such as BALFRAM [37], (3) explicitly showing the

effects of modelling assumptions on optimal tactical allocation strategies, (4) developing insights into optimal allocation strategies for various attrition structures of interest.

b. Thus, we are striving to provide a sound theoretical basis for application efforts such as by Chaiken. In our present work we have concentrated on solution techniques and methodology, since this was a gap in the current state-of-the-art for applications of optimal control/differential game theory to Lanchester-type models of warfare. We have studied idealizations (abstractions) of allocation structures arising in more complex (realistic) defense planning studies. Through such idealization and/or simplification we have tried to identify factors or concepts which merit further investigation in more detailed computer simulations (such as BALFRAM [37]). For example, uses have never exercised the "linear-law" attrition option of MILISTRAC and BALFRAM [36]. Yet our research indicates that optimal tactics with "linear-law" attrition are fundamentally different than those with "square-law" attrition. Hence, planners should be alerted (educated) to this fact about the dependence of optimal tactics on type of attrition process.

c. It seems appropriate to briefly discuss a modelling issue of abstraction versus realism. The mass of details present in a realistic (complex) model inevitably obscures essential modelling issues such as type of attrition process, type of objective (criterion) function. Our work is oriented towards clarifying basic issues by

examining simplified models. For example, versions of a well-known model (see pp. 96-104 in [72]) which are used for justifying tactics in the air war consider a combat payoff in terms of net sorties of ground support missions flown and totally ignore the effects of such sorties on the outcome of the ground war. Our theoretical analysis of this situation provides motivation (and justification) for a current ONR study [35] to determine the optimal tactics of the air war in the context of the ground war objectives for contingency plans of interest.

d. A sequential game arises in a current application of P. Chaiken's BALFRAM methodology [35]. Our research lays the theoretical foundations (via the research on differential games) for the inclusion of the capability to handle such a sequential game within the BALFRAM software system.

e. The questions of target priority rules for target selection frequently arises in the evaluation of proposed weapon systems and/or doctrine of employment for weapon systems. The study of idealized combat models would provide insight into the optimization of combat dynamics hopefully leading to a better quantitative basis for defense planning decisions.

f. In the Navy mission of fire support, the question of target priority rules for target selection frequently arises. An understanding of optimization principles in various dynamic combat situations would lead to better utilization of such resources.

7. Summary of Research Findings.

Here we summarize our research results. We found that solution techniques were insufficiently developed for the optimization of combat dynamics. Accordingly, this aspect received top priority in our work. Results are organized under the following headings:

(1) solution techniques,

(2) insights into optimal tactical allocation,

(3) implications for defense planning.

Items (2) and (3) differ in that the latter is a management-oriented digest of the practical implications of our research whereas the former is oriented towards a technical audience. Further amplification of results and conclusions is to be found in the appendices.

a. Solution Techniques.

Our research has produced the following results on solution techniques for the optimization of combat dynamics. Specifically, we have accomplished the following:

(1) developed general solution algorithm (based on results from modern optimal control theory) to determine the optimal control of deterministic Lanchester processes,

(2) developed the basic equations of optimality for the optimal control of the Lanchester stochastic process,

(3) developed a theory (theory of state variable inequality constraints) for the adequate treatment of non-negativity restrictions on force levels in optimal control problems for deterministic Lanchester processes,

(4) discovered that no adequate theory of state variable inequality contraints exists in current theories for differential games,

(5) shown how to treat the following difficult aspects of optimal control problems for deterministic Lanchester processes:

(a) singular solutions,

(b) state variable inequality constraints,

(c) multiple extremals (including dispersal surfaces and dominated returns),

(6)* developed complete solutions for optimal fire distribution tactics in one-against-two-target-types Lanchester combat for two cases:

(a) terminal control battle (fight-to-the-finish),

(b) prescribed duration battle,

(7)* developed partial solutions for optimal fire distribution tactics for a force engaged in Lanchester combat for the following cases:

(a) one-against-two-target-types combat,

(A) some special cases of variable attrition-rate coefficients,

(B) "linear-law" attrition of target types,

(C) bounded rates for changing the distribution of fire,

(D) Helmbold's general attrition model,

(E) stochastic attrition process,

(b) one-against-n-target-types combat,

(A) constant attrition-rate coefficients,

(B) some special cases of variable attrition-rate coefficients,

(8) developed solutions to homogeneous force models with variable attrition-rate coefficients.

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*In (6) and (7) the attrition process is deterministic and the attrition rate of a target type is proportional to only the number of firers unless otherwise noted.

b. Insights into Optimal Tactical Allocation.

Based on our study of the optimization of combat dynamics using the mathematical theories of differential games/optimal control we have reached the following conclusions:

(1) the structure of the optimal allocation policy for fire distribution/target selection depends upon model form; more specifically, the optimal policy depends on force levels, weapon system capabilities (as measured by Lanchester attrition-rate coefficients), the type of attrition process, the target acquisition process, the values placed upon surviving force types, and the termination conditions of combat,

(2) for deterministic Lanchester one-versus-n-target-types combat in which the attrition rate for each enemy target type is proportional to only the number of firers, the optimal fire distribution policy has the following properties (for linear utility of survivors and no replacements):

(a) fire is always concentrated on a single target type (which may change without the annihilation of the previous target type),

(b) it may depend indirectly upon force levels (depending upon how survivors are valued),

(c) when enemy survivors are valued in direct proportion to their kill capability (as measured by their Lanchester attrition-rate coefficient against the homogeneous force),

(A) for constant attrition-rate coefficients

(I) fire is always concentrated on the available enemy target type which has the largest product of attrition-rate coefficients (i.e. the largest $a_i b_i$ where a_i represents the kill capability of the friendly force against the enemy's ith target type and b_i represents that of the enemy's ith force type),

(II) fire is only shifted to a new target type after the annihilation of the previously available target type with the largest product of attri-rate coefficients (i.e. no change in fire distribution without the annihilation of a target type),

(B) for variable attrition-rate coefficients in the case when the ratio of enemy weapon system capabilities (as measured by Lanchester attrition-rate coefficients) is a constant for any two enemy weapon system types

(I) assuming that no target type is annihilated, all fire is always concentrated on the target type with the largest product of attrition-rate coefficients,

(II) additional assumptions are required to analyze the case when a target type is annihilated,

(d) target priorities are not directly influenced by replacement rates when these are added to the model,

(3) for deterministic Lanchester one-versus-two-target-types combat in which the attrition rate for each enemy target type is proportional to the product of the numbers of firers and target types, the optimal fire distribution policy has the following properties (for constant attrition-rate coefficients, linear utility of survivors, and no replacements):

(a) it depends directly upon force levels,

(b) ϕ^* (the optimal fraction of Y-fire directed at X_1) takes on one of three values: 0, 1, or $a_2/(a_1+a_2)$; thus, it is sometimes optimal to split one's fire,

(c) when enemy survivors are valued in direct proportion to their kill capability (as measured by their attrition-rate coefficients against the homogeneous force)

(A) the ranking of target priorities is never reversed over time,

(B) the optimal policy is given explicitly by

$$\phi^* = \begin{cases} 1 & \text{for } a_1b_1x_1 > a_2b_2x_2, \\ \dfrac{a_2}{a_1+a_2} & \text{for } a_1b_1x_1 = a_2b_2x_2, \\ 0 & \text{for } a_1b_1x_1 < a_2b_2x_2, \end{cases}$$

(d) the structure of the optimal policy is not sensitive to the type of attrition sustained by the homogeneous force,

(4) for deterministic Lanchester one-versus-two-target-types combat in which the attrition rate for each target type is proportional to only the number of firers with (a) constant attrition-rate coefficients, (b) linear utility of survivors, and (c) no replacements, the optimal fire distribution policy may be different in the following two cases:

(a) prescribed duration battle,

(b) fight-to-the-finish,

(This difference can only occur when for $a_1b_1 > a_2b_2$ enemy target type two is valued in greater proportion to its kill capability than target type one is.)

(5) for the stochastic version of the prescribed duration battle described in (4) above, the optimal fire distribution policy sometimes depends upon the force levels,

(6) for the battle described in (4) above, when there is an upper bound on the rate of changing one's distribution of fire (through command and control limitations), the structure of the optimal fire distribution policy is not appreciably changed; the following statements can be made about the optimal policy:

(a) the same conditions remain for it to be optimal to have $\phi^* = 0$ or 1 (although clearly ϕ^* may have to take on intermediate values),

(b) one starts to shift fire earlier (as measured by the rate of destruction of target type value, i.e. product of attrition-rate coefficient and marginal value of target type) than in the previous case when ϕ^* can instantaneously jump, for example, from 0 to 1,

(7) the optimal fire distribution policy does not appear to be the same for models with a deterministic attrition process and those with a stochastic attrition process.

c. Implications for Defense Planning.

In our research reported here we have studied abstractions (idealizations) of allocation structures that commonly occur in defense

planning studies. After studying these simplifications in order to obtain significant information about the structure of the optimal allocation strategies, we have reached the following conclusions (generalizations of phenomena that we have observed in simplified cases) as to considerations that should be brought to the attention of defense planners. These results should be kept in mind by practitioners who perform more detailed computer simulation studies.

(1) The nature of the attrition process has a significant effect upon optimal strategies. At present we feel that the highest priority should be given to educate planners to the fact that the type of optimal allocation strategy is dependent on the nature of the attrition model. This has the operational planning implication that studies should be run with both square-law attrition and then linear-law attrition. Past history shows that, for example, the linear-law attrition option has never been exercised in BALFRAM [36].

(2) Force levels do affect optimal strategies. Whether one "wins" (superiority) or "loses" (inferiority) affects optimal strategies.

(3) Even the nature of the scenario (terminal control n prescribed duration conflict) may affect optimal strategies. Thus, if one develops "good" tactics for a 90 day European campaign, such tactics need not be "good" if the conflict does not terminate at the prescribed time.

(4) Optimal tactics for the air war are dependent upon whether or not the air war is evaluated in the context of ground war objectives.

(5) Optimal tactics are also significantly influenced by the nature of the target acquisition process and command and control capabilities.

8. Suggested Future Research Tasks.

After performing the research documented in this report, we feel that the current state-of-the-art for applying optimal control/ differential game theory to determine optimal allocation strategies for deterministic Lanchester processes is such that more significant results may be readily obtained in the future. We do not feel that previous applied research has been on a sound theoretical basis, and we hope that our work here has established that basis. With the development of a theory of state variable inequality constraints (SVIC's) for differential games we feel that the capabilities for solving applied problems would be appreciably increased. Therefore, we give this task top priority.

Based on our past research experience we feel that there is much to be accomplished in the future. Specifically, we suggest the following as future research tasks:

(a) Development of a theory of state variable inequality constraints for Lanchester-type differential games.

(b) Development of further results for deterministic Lanchester one-versus-n-target-types combat. This work would build upon past research results and would include

  (1) development of more complete solutions (for cases of both constant attrition-rate coefficients and also variable attrition-rate coefficients),

  (2) incorporation into the models of such additional factors as replacements, logistics, and supporting weapons,

  (3) further consideration of "linear-law" attrition of target types.

(c) Study of the optimal division of forces to engage target types. A slightly different attrition model than in the Isbell-Marlow fire distribution problem is one in which the friendly forces (assumed homogeneous) only suffer casualties when they engage an enemy target type. (In the Isbell-Marlow problem the friendly forces sustain casualties from an enemy target type regardless of whether or not they engage it.) This model is especially important, since it can be used to answer several questions that have arisen in the work of P. Chaiken.

(d) Development of simple (probably non-optimal) fire distribution policies as approximations to the optimal policy. The worth of such an approximate optimal policy would be evaluated by comparing the payoff associated with it to the optimal return. Besides the intrinsic importance of this aspect, we suggest this task, since we have developed complete solutions to several deterministic Lanchester one-versus-two-target-type combat problems.

(e) Examination of other one-sided allocation problems of tactical interest in the Lanchester theory of combat (for example, the optimal distribution of supporting fires (such as Naval fire support) over enemy units in Lanchester combat with friendly forces). Emphasis would be on solution development for each of various modelling alternatives in order to study the dependence of optimal tactics on model form.

(f) Performance of computational studies on the optimal control of the Lanchester stochastic process. Using the finite difference approximation developed in Appendix I, we would compare optimal strategies for deterministic and stochastic Lanchester processes. Furthermore, we would develop the fundamental functional equation for the terminal control battle (in Appendix I we consider the prescribed duration battle) and again make such a comparison. Theoretical investigations should concentrate on gaining more insight into the structure of the optimal target selection policy when the attrition mechanism is stochastic.

(g) Further study of the optimal control of deterministic Lanchester processes when there is a bound on the rate of change of the distribution of fire. Results of our initial effort on this topic are contained in Appendix J. We suggest extension of these preliminary results.

(h) Further study of the dependence of the optimal fire distribution policy on the nature of the attrition model. We have given some preliminary results in Appendix K for Helmbold's general attrition structure. We suggest further extension of this work and consideration of still other attrition models (for example, that of Willard [131]).

(i) Examination of optimal tactics for other objective functions (i.e. performance (effectiveness) criteria). For all the work contained in this report the objective function has been a linear function of the number of survivors. Other objective functions that might be considered are

(1) nonlinear valuation of survivors (such as quadratic as motivated by Lanchester's square law),

(2) the value of losses rather than survivors.

(j) Consideration of some simple Lanchester-type differential games. All previous work has been handicapped by the lack of an adequate theory of SVIC's for differential games. Such a theory can be used to reduce the volume of allocation strategy combinations that must be considered. Previous work in the literature for which the non-negativity restrictions on the force levels has not received adequate treatment is as follows:

(1) the war of attrition and attack of R. Isaacs [72],

(2) the supporting weapon system game of H. K. Weiss [130],

(3) the logistics allocation game of Moglewer and Payne [95],

(4) the "two-on-two" differential game studied by S. Sternberg [111].

After the development of a theory of SVIC's for Lanchester differential games, it would seem appropriate to re-examine the above work. Results from studying these problems are useful to the work of P. Chaiken.

(k) Study of the optimal tactics for the air war in the case when the air war is evaluated in the context of ground war objectives.

Further details on proposed future work is to be found in the discussion sections of Appendices E, G, I, J, and K.

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Appendix A. On the Isbell and Marlow Fire Programming Problem.

1. Introduction.

In this appendix a complete solution is derived to the Isbell and Marlow fire programming problem. The original work of Isbell and Marlow has been extended by determining the regions of the initial state space from which optimal paths lead to each of the terminal states of combat. The solution process has involved determining the domain of controllability for each of the terminal states of combat and the determination of dispersal surfaces. This solution process suggests a solution procedure applicable to a wider class of tactical allocation problems, terminal control attrition differential games. The structure of optimal target engagement policies in "fights to the finish" is discussed.

An underdeveloped area [6] of the Lanchester theory of combat is target selection for combat among heterogeneous forces. This type of problem has been studied by Isbell and Marlow, who considered both a truncated stochastic (Lanchester) process by game theoretic means [12] and a terminal control (one-sided) differential game [11]. An attrition differential game is an idealized combat situation described by Lanchester-type equations [5], [18] over a period of time with choices of tactics available to both sides and subject to change with time. Terminal control attrition games only end when the course of combat has been steered to a prescribed state.

In developing a theory of target selection it is important to understand the dependence of allocation rules on the type of model chosen. Tactical allocation problems may be studied in two types of scenarios: (1) the prescribed duration battle and (2) the terminal control battle (a particular case of which is the "fight to the finish"). All the attrition examples in Isaacs' book [14] are of the first type (his "War of Attrition and Attack" is the continuous version of the tactical air war game [3], [4], [9] studied at RAND). Only Isbell and Marlow [11] and Weiss [19] have studied the terminal control problem. Unfortunately, Isbell and Marlow did not obtain a complete solution to their problem. They could not determine when certain terminal states of combat were reached. Weiss studied a problem which may be considered to be a generalization (two-sided version) of their problem. Writing many years before results were known beyond a small number of researchers, he did not use the subsequently well-known necessary conditions [1]. That his solution procedure [19] was heuristic is not surprising, since the simpler problem* [11] which he referenced in his paper had not been completely solved.

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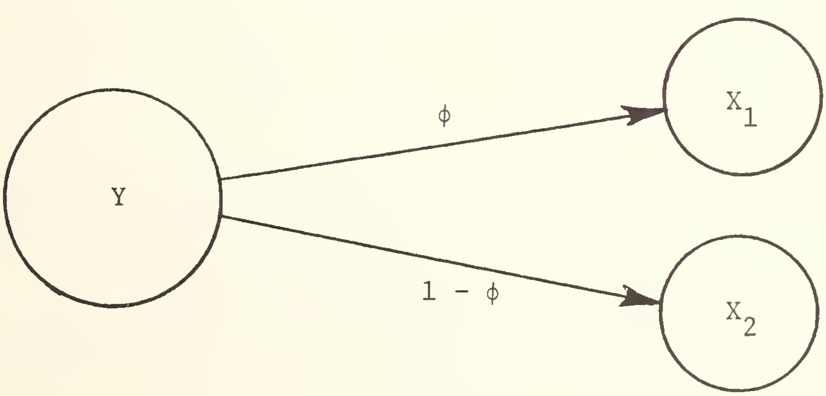
*Since the original version of this paper, this problem has also been recently studied by S. Sternberg [17]. He employed a more geometric approach closely following that of R. Isaacs [14] and originally employed by Isbell and Marlow [11]. Sternberg obtained different results than those presented here.

Thus the obtaining of a complete solution to the Isbell and Marlow problem is viewed as a first step leading to the solution of more general tactical allocation problems. In this present appendix, we shall derive its solution by the Pontryagin maximum principle [16]. Generalizations of this procedure to differential games are indicated, and solution properties are discussed. In view of the close connection [1], [10] between optimal control and differential games (Isaacs), the terminology of these two fields is used somewhat interchangeably.

2. The Fire Distribution Problem.

The situation considered by Isbell and Marlow [11] is the simplest problem of fire distribution: combat between an X-force at two force types (for example, riflemen and grenadiers) and a homogeneous Y-force (for example, riflemen only). This situation is shown diagrammatically below.



It is the objective of the Y-force commander to maximize his survivors at the end of battle and minimize those of his opponent (considering the utilities assigned survivors). This is accomplished through his choice of the fraction for fire, ϕ, directed at X_1. The battle terminates when one side or the other has been annihilated.

Mathematically the problem may be stated as

$$\underset{\phi(t)}{\text{maximize}}\ ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ unspecified}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2 \tag{1}$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where all symbols are defined in the next section and with terminal states defined by (1) $x_1(T) = x_2(T) = 0$ and (2) $y(T) = 0$.

The terminal surface of the "realistic" (one-sided) game is seen to consist of five parts:

C_1: $x_1(T) = 0$, $x_2(T) > 0$, $y(T) = 0$

C_2: $x_1(T) = 0$ before $x_2(T) = 0$, $y(T) > 0$,

C_3: $x_1(T) = 0$ after $x_2(T) = 0$, $y(T) > 0$,

C_4: $x_1(T) > 0$, $x_2(T) = 0$, $y(T) = 0$,

C_5: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$.

3. Notation.

The symbols which are used in this paper are defined as follows:

$A = A(R,z) = [z^2(R-1) - R]/(z-1)^2$,

$B = B(R,z) = A(z-1)^2/z^2 = [z^2(R-1) - R]/z^2$,

a_1, a_2, b_1, b_2 = constant attrition rates,

C_i for $i = 1,2,3,4,5$ = the $i^{\underline{th}}$ part of the terminal surface as defined in section 2.,

$D(C_i)$ = domain of controllability for C_i,

$g(P^o,R,z)$ = term in equation (57) of the locus of points for which $P_1 = P_4$,

$h(P^o,R,z)$ = term in equation (58) for boundary surface between the regions from which optimal paths lead to C_1 and C_4,

p,q,r = utilities assigned to surviving X_1, X_2 and Y forces respectively,

$p_i(t)$ for $i = 1,2,3$ = dual variable corresponding to $x_i(t)$ $(x_3(t) = y(t))$,

p_i^o for $i = 1,2,3$ = boundary condition for dual variable at $\tau = 0$,

P_i for $i = 1,2,3,4,5$ = payoff associated with an extremal leading to C_i,

$P^o = (x_1^o, x_2^o, y_0)$ = point in the initial state space,

$R = a_1 b_1/(a_2 b_2)$,

$s = s(x_1^o, x_2^o) = b_1 x_1^o + b_2 x_2^o$,

t_1 = time at which X_1 is annihilated, i.e. $x_1(t_1) = 0$,

t_2 = first time at which $2b_1 x_1(t_2) x_2^o + b_2 (x_2^o)^2 = a_2 y^2(t_2)$ for an extremal leading to C_4,

T = total time for the battle,

$v = v(\tau) = a_2 p_2(\tau) - a_1 p_1(\tau)$,

$w = \cosh \sqrt{a_2 b_2}\, \tau_1(C_4) = \dfrac{a_1}{p_2^o} \dfrac{(b_1 p_2^o + b_2 p)}{(a_1 b_1 - a_2 b_2)}$,

x_1, x_2, y = average force strengths; with initial values x_1^o, x_2^o, y_o,

$z = \cosh \sqrt{a_2 b_2}\, \tau_1(C_5) = \dfrac{a_1}{q} \dfrac{(b_1 q - b_2 p)}{(a_1 b_1 - a_2 b_2)} = \dfrac{R - \delta}{R - 1}$,

$\delta = a_1 p/(a_2 q)$,

ϕ = fraction of Y-fire directed at X_1,

τ = "backwards time" from the end of the reduced game, i.e. the time remaining before the end of the reduced game,

$\tau_1(C_i)$ = "backwards time" of the first switch in tactics for extremals leading to C_i,

$\hat{\tau}_1 = T - t_1$ = length of time that Y fires at X_2 after X_1 has been annihilated for C_1 extremals.

4. Solution Procedure and Extensions.

Extremal paths (a path on which the necessary conditions for optimality are almost everywhere satisfied) may be obtained by routine application of Pontryagin's maximum principle [16] (the original

authors used equivalent conditions independently developed by Isaacs [13]). However, in a terminal control problem we would like to know the domain of controllability [7] for each terminal state so that tactics are determined in terms of the initial conditions of combat (and also possibly time). We define the domain of controllability for a given terminal state to be that subset of the initial state space from which extremals lead to the terminal state. Furthermore, additional considerations (as outlined below) are required when the domains of controllability corresponding to different terminal states overlap.

The following procedure has been used to solve the above problem:

(a) extremal control is determined by maximizing the Hamiltonian; since the state variables (force strengths) are non-negative, the control depends, in many cases, only on relationships between the dual variables (marginal return from destroying target),

(b) from each separate terminal state, the time history of the dual variables is obtained by a backward integration of the adjoint system of differential equations; for a square law attrition process, the adjoint equations are independent of the state variables,

(c) for each terminal state the domain of controllability is determined by forward integration of the state equations using the time history of extremal control developed in (b); changes in control with time (existence of transition surface) may have to be considered in this step.

(d) the solution is determined at this point for regions of the initial state space which are covered by part of the domain of controllability for only one terminal state; one must also verify that the entire initial state space has been accounted for, since otherwise one may have overlooked some type of "singular" surface or even a terminal state,

(e) if domains of controllability overlap so that for a point of the initial state space contained in their intersection there is more than one extremal leading to the terminal surface, then one computes the payoff associated with each extremal; the optimal trajectory is selected from the extremals by comparing these values.

It is noted that Isbell and Marlow [11] stopped at step (b) above.

It seems appropriate to further discuss the situation when, for example, the domains of controllability corresponding to two different terminal states overlap. For points of the initial state space contained in the intersection of the two domains, there are two mutually exclusive possibilities: (1) for every point the payoff associated with the extremal leading to one terminal state always dominates that corresponding to the other, or (2) the payoff corresponding to extremals of one family does not always dominate that of the other. In the first case, the solution is simply determined: the optimal trajectories are the extremals corresponding to the more favorable payoff. In the second case, one must determine for what

parts of the region of intersection each alternative is better. For the problem at hand, it will be seen that such a region of intersection may be divided into two subregions separated by a boundary surface. In each subregion the payoff corresponding to one family of extremals dominates that of the other. R. Isaacs (see pp. 132-141 of [14]) has referred to such a singular surface as a dispersal surface, since when the geometry of the optimal trajectories is considered, optimal paths only lead away from a dispersal surface. For the problem at hand, it will be seen that for a certain range of model parameters, two dispersal surfaces are present in its solution.

The complete solution to this problem is shown in Tables I and II with supporting details presented in the next section. When $a_1p \geq a_2q$, the domains of controllability do not overlap so that the extremals are unique, and the extremal control turns out to be the optimal control. Hence, the solution procedure terminates at step (d) above. When $a_2q > a_1p$, the domains of controllability are shown in Table I as Case (2). For $R - \sqrt{R(R-1)} \leq \delta < 1$ where $\delta = (a_1p)/(a_2q)$, the domains of controllability don't overlap except for that corresponding to part C_4 of the terminal surface, which overlaps several others. In section 5.2 below it is shown how optimal trajectories are determined in the region of overlap and that extremals leading to C_4 can only be optimal for $a_1b_1y_o{}^2 \leq s^2 + A(b_2x_2^o)^2$. The resulting solution is shown in Table II as Case (a). It should be noted that non-binding constraints, which define the regions in the initial state space for

TABLE I. Solution to Target Selection Problem - Fight to Finish

Nonrestrictive assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$

Terminal State	Optimal Control	Domain of Controllability
Case (1): $a_2q \le a_1p$		
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 \le t \le T \end{cases}$	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 \ge s^2 - (b_2x_2^o)^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 \le t \le T \end{cases}$	$a_1b_1y_o{}^2 > s^2 + (R-1)(b_2x_2^o)^2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 1$ for $0 \le t \le T$	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 < s^2 - (b_2x_2^o)^2$
	Extremal Control	
Case (2): $a_2q > a_1p$		
C_1	Same as Case (1)	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 \ge s^2 + B(b_2x_2^o)^2$
C_2	Same as Case (1)	Same as Case (1)
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 \le t \le T \end{cases}$	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 \ge R\{s^2 - (b_1x_1^o)^2\}$ $a_1b_1y_o{}^2 \le s^2 + R(R-1)(b_2x_2^o)^2$
C_5	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 < R\{s^2 - (b_1x_1^o)^2\}$ $a_1b_1y_o{}^2 \le Rs^2\{1 - \frac{1}{z^2}\}$
C_5	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T - \tau_1 \\ 0 & \text{for } T - \tau_1 \le t \le T \end{cases}$	$a_1b_1y_o{}^2 < s^2 + (R-1)(b_2x_2^o)^2$ $a_1b_1y_o{}^2 < s^2 + B(b_2x_2^o)^2$ $a_1b_1y_o{}^2 > s^2 + A(b_2x_2^o)^2$ $a_1b_1y_o{}^2 > Rs^2\{1 - \frac{1}{z^2}\}$

Definition of Times

(a) t_1 is first t such that $x_1(t_1) = 0$.

(b) t_2 is first t such that $2b_1x_1(t_2)x_2^o + b_2(x_2^o)^2 = a_2y^2(t_2)$.

(c) τ_1 is determined by $\cosh \sqrt{a_2b_2}\,\tau_1 = \frac{a_1}{q}\frac{(b_1q-b_2p)}{(a_1b_1-a_2b_2)}$.

TABLE II. Target Selection Problem - Fight to Finish

Solution for $a_2 q > a_1 p$ and Various Values of Parameters

Nonrestrictive assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (a): $A \leq 0$, i.e. $R - \sqrt{R(R-1)} \leq \delta < 1$

	Terminal State	Optimal Control	Region of Initial Force Levels
C_1	$\begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \\ 0 & \text{for } t_1 \leq t \leq T \end{cases}$	$a_1 b_1 y_o{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 \geq s^2 + B(b_2 x_2^o)^2$
C_2	$\begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \\ 0 & \text{for } t_1 \leq t \leq T \end{cases}$	$a_1 b_1 y_o{}^2 > s^2 + (R-1)(b_2 x_2^o)^2$
C_4	$\begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_2 \\ 0 & \text{for } t_2 \leq t \leq T \end{cases}$	$a_1 b_1 y_o{}^2 \geq R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_o{}^2 \leq s^2 + A(b_2 x_2^o)^2$
C_5	$\begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T - \tau_1 \\ 0 & \text{for } T - \tau_1 \leq t \leq T \end{cases}$	$a_1 b_1 y_o{}^2 > s^2 + A(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 < s^2 + B(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 > Rs^2\{1 - 1/z^2\}$
C_5		$\phi^*(t) = 0$ for $0 \leq t \leq T$	$a_1 b_1 y_1{}^2 \leq Rs^2\{1 - 1/z^2\}$ $a_1 b_1 y_o{}^2 < R\{s^2 - (b_1 x_1^o)^2\}$

Case (b): $A > 0$, i.e. $0 \leq \delta < R - \sqrt{R(R-1)}$

	Terminal State / Optimal Control	Region of Initial Force Levels
C_1	Same as Case (a)	$a_1 b_1 y_o{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 \geq Rs^2 - R\{b_1 x_1^o[z^2(R-1)+R]/(2R) + b_2 x_2^o\}^2/z^2$ for $0 \leq x_1^o < (b_2 x_2^o)/(kb_1)$ $a_1 b_1 y_o{}^2 \geq h(P^o, R, z)$ for $0 \leq x_2^o \leq kb_1 x_1^o/b_2$
C_2	Same as Case (a)	Same as Case (a)
C_4	Same as Case (a)	$a_1 b_1 y_o{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 \geq R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_o{}^2 \leq h(P^o, R, z)$ for $0 \leq x_2^o \leq kb_1 x_1^o/b_2$
C_5	$\phi^*(t) = 0$ for $0 \leq t \leq T$	$a_1 b_1 y_o{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_o{}^2 < R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_o{}^2 \leq Rs^2 - R\{b_1 x_1^o[z^2(R-1)+R]/(2R) + b_2 x_2^o\}^2/z^2$ for $0 \leq x_1^o < (b_2 x_2^o)/(kb_1)$

Explanation of Symbols

(1) For t_1, t_2 and τ_1, see Table I.

(2) $k = [z^2 - R(z-1)^2]/(2R)$.

the various types of optimal control, have been deleted in constructing Case (a) of Table II from Case (2) of Table I. For $0 \leq \delta < R - \sqrt{R(R-1)}$, $D(C_1)$, the domain of controllability for the C_1 part of the terminal surface, and $D(C_5)$ have a non-empty intersection and similarly for $D(C_1)$ and $D(C_4)$. Hence, step (e) in our above solution procedure must be applied. In section 5.3 the determination of the two dispersal surfaces is shown. The resulting solution is shown in Table II as Case (b).

The above solution procedure may be easily extended to terminal control differential games (such as [19] in which the subsequently well-known necessary conditions [1] were not applied). Moreover, in two-sided problems (both the X-force and the Y-force are heterogeneous and each is faced with a fire distribution problem affected by the other's choice) this author has noted that not only may domains of controllability overlap with there being multiple extremals from a given point in the initial state space, but also the corresponding terminal states imply that the same player may either win or lose depending upon which extremal path is followed. This is illusory, however, since the player's opponent may block his steering the course of battle to a winning end point by use of a non-extremal strategy. Hence, the player must always lose.

We should note that in the existing theory of differential games [2], [14] it is assumed that the optimal strategies are pure. Since the Hamiltonian is separable (a function independent of the X-force's strategy variable(s) plus a function independent of the

Y-force's strategy variable(s)) for two-sided fire distribution problems in the Lanchester theory of combat as formulated by Isbell and Marlow [11] and Weiss (see pp. 94-95 of [18]), it may be shown that there do exist pure strategy solutions to such problems [8]. Thus, this paper's developments are applicable to this class of problems.

5. Development of Solution.

The solution is actually derived for a "reduced" game (that portion of battle during which Y is faced with a choice problem). We illustrate here for extremals to C_1. It suffices to trace extremals up to t_1 when $x_1(t_1) = 0$, since $\phi^* = 0$ from then until the end of the game. The determination of the value, denoted by $V(x_1,x_2,y)$, of the reduced game, which is needed to determine the values of the adjoint variables on the terminal surface, and part of the solution originally obtained by Isbell and Marlow will not be repeated here although we shall outline the general steps.

The Hamiltonian is

$$H(t,x_i,p_i,\phi) = -\{p_1\phi a_1 y + p_2(1-\phi)a_2 y + p_3(b_1x_1 + b_2x_2)\}, \quad (2)$$

and the adjoint equations are

$$\frac{dp_1}{dt} = b_1p_3,$$

$$\frac{dp_2}{dt} = b_2p_3,$$

$$\frac{dp_3}{dt} = p_1a_1\phi + p_2(1-\phi)a_2, \quad (3)$$

with

$$p_1(t=t_1) = \text{unspecified}$$

$$p_2(t=t_1) = \frac{\partial V}{\partial x_2} = \frac{-q\sqrt{b_1 x_2}}{\sqrt{b_2 x_2^2 - a_2 y^2}}$$

$$p_3(t=t_1) = \frac{\partial V}{\partial y} = \frac{q a_2 y}{\sqrt{b_2}\,\sqrt{b_2 x_2^2 - a_2 y^2}} \tag{4}$$

The extremal control ϕ^* is determined by the maximum principle, i.e. $\underset{0\le\phi\le 1}{\text{maximum}}\ H(t,x_i,p_i,\phi)$, and we also have that

$$H(t,x_i,p_i,\phi^*) = 0. \tag{5}$$

Equations (4) and (5) may be combined to yield that for extremals leading to C_1 we have

$$p_1(t=t_1^-) = \frac{-q a_2\sqrt{b_2}\ x_2(t=t_1)}{a_1\sqrt{b_2 x_2^2 - a_2 y^2}}, \tag{6}$$

or

$$p_1(t=t_1^-) = \frac{a_2}{a_1}\, p_2(t_1), \tag{7}$$

where

$$p_1(t=t_1^-) = \lim_{\substack{t\to t_1\\ t\le t_1}} p_1(t).$$

Obtaining a solution to this problem is simplified by the following considerations. Let $\tau = t_1 - t$ and define

$$v(\tau) = a_2 p_2(\tau) - a_1 p_1(\tau), \tag{8}$$

then we have

$$\frac{dv}{d\tau} = (a_1b_1 - a_2b_2)p_3(\tau), \qquad (9)$$

with

$$v(\tau=0) = a_2p_2(\tau=0) - a_1p_1(\tau=0), \qquad (10)$$

and where (up until the first shift of tactics)

$$p_3(\tau) = p_3(\tau=0)\cosh\{\sqrt{\phi a_1b_1 + (1-\phi)a_2b_2}\ \tau\}$$

$$- \frac{\phi a_1p_1(\tau=0)+(1-\phi)a_2p_2(\tau=0)}{\sqrt{\phi a_1b_1 + (1-\phi)a_2b_2}} \sinh\{\sqrt{\phi a_1b_1 + (1-\phi)a_2b_2}\ \tau\} . \qquad (11)$$

It is easily seen that $p_3(\tau) > 0$ for all τ, and thus

$$\frac{dv}{d\tau} > 0 \quad \text{for all} \quad \tau, \qquad (12)$$

when $a_1b_1 > a_2b_2$ by (9). As determined by the maximim principle and use of (2) and (8), the extremal control is given by

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad v(\tau) > 0 \\ 0 & \text{for} \quad v(\tau) < 0. \end{cases} \qquad (13)$$

It is easy to show that it is impossible for $v(\tau) = 0$ over any finite interval of time, and hence the possibility for any singular solution [15] to this problem is excluded. (The reader should note that the term "singular surface" is used in a slightly different sense by Isaacs (see pp. 132-134 and pp. 156-158 of [14]) and in the control theory literature (see p. 502 of [10] or p. 4 of [15]).)

By the symmetry of the problem it suffices to assume that $a_1b_1 > a_2b_2$, and then equations (7), (8), (12) and (13) imply that extremals can actually lead to C_1. Furthermore, for $a_1b_1 > a_2b_2$ it is readily shown that no extremals lead to C_3 (i.e. $D(C_3)$ is void) and that extremals can only lead to C_4 when $x_2(t) > 0$ for $t < T$. We shall sketch the proof that no extremals for which $x_2(t) = 0$ for $t < T$ can lead to C_4. By arguments similar to those used above for C_1 and using the value of the reduced game $V(x_1,x_2,y)$ and (2) and (5) with $\tau = 0$ and $\phi^* = 0$, it is readily shown that $v(\tau=0^+) = 0$ where τ is the backwards time from the end of the reduce game, i.e. $x_2(\tau=0) = 0$, and $v(\tau=0^+)$ is a one-sided limit through positive values. Considering (12), this implies that $v(\tau) > 0$ for $\tau > 0$, and hence by (13) X_2 is never fired at so that C_4 cannot be reached. The special case in which C_4 is reached but $x_2(t) > 0$ for $t < T$ is discussed in section 5.2.

In the next three subsections we present additional details of solution development: (1) determination of domains of controllability, (2) entry to C_4, and (3) determination of dispersal surfaces. This material represents the extension of the original work of Isbell and Marlow [11].

5.1. <u>Determination of Domains of Controllability</u>.

One contribution of this note is to show how to determine the domains of controllability. There are two cases to consider.

Case (1) $a_2q \le a_1p$

These determinations are routine and consist in combining the time history of the extremal control, the non-negativity requirements of the state variables, and the generalized square law

$$u^2(t_1) - u^2(t_2) = \{\phi a_1 b_1 + (1-\phi)a_2 b_2\}(y^2(t_1) - y^2(t_2)), \quad (14)$$

where $\phi(t) = \text{const.}$ in $t_1 \le t \le t_2$ and $u(t) = b_1x_1(t) + b_2x_2(t)$.

Case (2) $a_2q > a_1p$

The condition for entry to C_2 is as before. There are two subcases for entry to C_5. First we consider the extremals for which

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T - \tau_1 \\ 0 & \text{for } T - \tau_1 \le t \le T. \end{cases} \quad (15)$$

Combining the above extremal control with the generalized square law (14), we obtain

$$R(b_2x_2(T))^2 + 2Rb_1b_2x_1(T-\tau_1)x_2(T) + (b_1x_1(T-\tau_1))^2$$
$$+ 2(1-R)b_1b_2x_1(T-\tau_1)x_2^o + a_1b_1y_o{}^2 - s^2 + (1-R)(b_2x_2^o)^2 = 0. \quad (16)$$

A backwards integration of the state equations of $0 \le \tau \le \tau_1$ with $x_2(\tau=0) = x_2(T)$ and $y(\tau=0) = 0$ yields that

$$b_2x_2^o = b_2x_2(T)z + b_1x_1(T-\tau_1)(z-1), \quad (17)$$

since $x_2(t=T-\tau_1) = x_2^o$. It should be noted that in deriving (17) we have assumed that $x_1(T-\tau_1) > 0$. (16) and (17) may be combined to yield

$$(b_2x_2(T)-b_2x_2^o)^2 = (a_1b_1y_o{}^2-s^2)/A, \tag{18}$$

and hence we require that the right-hand side of (18) be non-negative in order that C_5 be reached. It may be shown that $A > 0$ leads to a contradiction to a result (32) derived below,* and hence $A \leq 0$ for extremals corresponding to (15) to lead to C_5. This corresponds to $R - \sqrt{R(R-1)} \leq \delta < 1$, since $A = [z^2(R-1) - R]/z^2$ and $z = (R-\delta)/(R-1)$. When $A \leq 0$, solution of (18) for $x_2(T)$ and the requirement that $x_2(T) > 0$ leads to

$$a_1b_1y_o{}^2 > s^2 + A(b_2x_2^o)^2. \tag{19}$$

Combination of the expression for $x_2(T)$ with (17) and the requirement that $x_1(T-\tau_1) > 0$ leads to

$$a_1b_1y_o{}^2 < s^2 + (b_2x_2^o)^2A(z-1)^2/z^2. \tag{20}$$

Next, we consider the extremals for which

$$\varphi^*(t) = 0 \quad \text{for} \quad 0 \leq t \leq T. \tag{21}$$

It is readily shown that

$$y(t) = y_o \cosh\sqrt{a_2b_2}\, t - \frac{(b_1x_1^o + b_2x_2^o)}{\sqrt{a_2b_2}} \sinh\sqrt{a_2b_2}\, t, \tag{22}$$

so that the boundary, which occurs when $y(T) = 0$ for $T = \tau_1$, between this case and the previous one is easily seen to be given by

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*There is a flaw in this argument. The final solution, however, is not affected by this. See Appendix F for a further elaboration of this and other subtle points.

$$y_o^{\,2}[\cosh\sqrt{a_2b_2}\ \tau_1]^2 = \frac{(b_1x_1^o+b_2x_2^o)^2}{a_2b_2}\{[\cosh\sqrt{a_2b_2}\ \tau_1]^2 - 1\}, \quad (23)$$

where the τ-time of the switch, determined from integration of the adjoint equations and the maximum principle, is given by

$$z = \cosh\sqrt{a_2b_2}\ \tau_1 = \frac{a_1}{q}\,\frac{(b_1q - b_2p)}{(a_1b_1 - a_2b_2)}\ . \quad (24)$$

Noting that $\phi^* = 0$ for the entire battle when $T < \tau_1$ and re-arranging, we obtain the result shown in Table I. Combination of the extremal control (21) with (14) also yields a quadratic equation for $b_2x_2(T)$ which may be solved to yield

$$b_2x_2(T) = -b_1x_1^o + \sqrt{s^2 - a_2b_2y_o^{\,2}}. \quad (25)$$

Thus, the payoff for extremals on which (21) holds is given by

$$P_5 = \frac{q}{b_2}\{(b_1x_1^o)\frac{(R-\delta)}{R} - \sqrt{s^2 - (a_1b_1y_o^{\,2})/R}\}. \quad (26)$$

The requirement that $x_2(T) > 0$ leads to the result shown in Table I. When this latter condition or (19) is violated, then it is not possible for $x_2(T) > 0$ when $\phi^*(\tau) = 0$ for $0 \leq \tau \leq \tau_1$. Hence, we must investigate the possibility that $x_2(T) = 0$ (C_4 is entered), and this is done in the next section.

Finally, we consider extremals leading to C_1. As done above, we may combine the extremal control with (14) and the requirement that $x_2(T) > 0$ to obtain the first result shown in Table I. Similarly, $y(t_1) \geq 0$ implies that

$$a_1 b_1 y_o^{\ 2} \geq (b_1 x_1^o)^2 + 2b_1 b_2 x_1^o x_2^o. \tag{27}$$

Also, the payoff for extremals leading to C_1 is given by

$$P_1 = -\frac{q}{\sqrt{R}\, b_2} \sqrt{s^2 + (R-1)(b_2 x_2^o)^2 - a_1 b_1 y_o^{\ 2}}. \tag{28}$$

It remains to develop conditions which distinguish between optimal paths leading to C_1 and C_5. (Thus, strictly speaking the result for C_1 shown in Table I Case (2) is for the region where extremals of $D(C_1)$ yield a larger payoff than extremals to C_5 when the two fields of extremals overlap.) We do this for $A < 0$, i.e. $z^2 < R/(R-1)$, since different arguments are required to determine optimal trajectories when $A > 0$ (see section 5.3). It will be shown that optimal paths can only lead to C_1 when the time remaining in the battle after destruction of the X_1 forces exceeds the maximum time to fire at X_2 for extremals reaching C_5.

Let $\hat{\tau}_1 = T - t_1$ denote the length of time that Y fires at X_2 after X_1 has been annihilated. Then optimal paths can only reach C_1 when $\hat{\tau}_1 \geq \tau_1(C_5) = \tau_1$, where $\tau_1(C_5)$ is the "backwards time" of the switch in tactics for C_5 extremals. First, we shall show that extremals from P^o with $x_1^o > 0$ leading to C_5 yield a better payoff than those leading to C_1 for $a_1 b_1 y_o^{\ 2} \leq Rs^2\{1 - 1/z^2\}$ and $z^2 < R/(R-1)$. Thus, optimal paths can only lead to C_1 from initial points for which $a_1 b_1 y_o^{\ 2} > Rs^2\{1 - 1/z^2\}$. However, for such points the extremal strategy is to use $\varphi^*(t=0) = 1$ and to continue this until either the switching surface is encountered or X_1 is

annihilated, i.e. $\phi^*(t) = 1$ for $0 \leq t \leq$ minimum $(T-\tau_1,\ T-\hat{\tau}_1)$. Hence, optimal paths can only reach C_1 when $\hat{\tau}_1 \geq \tau_1$.

Thus, it remains to sketch the proof that extremals from P^o with $x_1^o > 0$ leading to C_5 yield a larger payoff than those leading to C_1 for $a_1b_1y_o{}^2 \leq Rs^2\{1 - 1/z^2\}$ and $z^2 < R/(R-1)$. Setting $a_1b_1y_o{}^2 = Rs^2\{1 - 1/z^2\}$ in the expressions for P_5 and P_1 (equations (26) and (28), respectively), it is easily shown that $P_5 > P_1$ for $z^2 < R/(R-1)$. Then since $\frac{\partial P_1}{\partial y_o} \geq \frac{\partial P_5}{\partial y_o} \geq 0$, it is readily seen that $P_5 > P_1$ for all y_o such that $a_1b_1y_o{}^2 \leq Rs^2\{1 - 1/z^2\}$ when $A < 0$.

Combining the extremal control for C_1 with (14), we have that

$$y^2(t_1) = \frac{1}{a_1b_1}\{a_1b_1y_o{}^2 - (b_1x_1^o)^2 - 2b_1b_2x_1^ox_2^o\}. \tag{29}$$

Also, since $\phi^*(t) = 0$ for $t_1 \leq t \leq T$, integration of the state equations (1) yields that

$$y(t) = y(t{=}t_1)\cosh\sqrt{a_2b_2}\,(t-t_1) - x_2^o\sqrt{\frac{b_2}{a_2}}\sinh\sqrt{a_2b_2}\,(t-t_1). \tag{30}$$

Since $y(t{=}T) = 0$, we have that

$$\tanh\sqrt{a_2b_2}\,\tau_1 = \frac{y(t{=}t_1)}{x_2^o}\sqrt{\frac{a_2}{b_2}}. \tag{31}$$

Since the hyperbolic cosine is a strictly increasing function of its argument, $\hat{\tau}_1 \geq \tau_1$ implies that $\cosh\sqrt{a_2b_2}\,\hat{\tau}_1 \geq \cosh\sqrt{a_2b_2}\,\tau_1 \geq 1$. Combination of this with (29) and (31), use of well-known identities for the hyperbolic functions, and some algebraic manipulation yields that

$$a_1 b_1 y_o^{\;2} \geq (b_1 x_1^o + b_2 x_2^o)^2 + \left\{\frac{z^2(R-1) - R}{z^2}\right\} (b_2 x_2^o)^2. \tag{32}$$

For $1 \leq z^2 \leq R/(R-1)$, it is easily seen that (32) implies (27) so that this latter condition need not be considered further.

5.2. Entry to Terminal State C_4.

Terminal state C_4 is reached whenever X_2 can be annihilated by Y during the battle's last phase of length τ_1. The boundary conditions on the terminal surface are

$$\begin{array}{ll} x_1(T) > 0 & p_1(\tau=0) = -p \\ x_2(T) = 0 & p_2(\tau=0) = p_2^o, \text{ unspecified} \\ y(T) = 0 & p_3(\tau=0) = p_3^o, \text{ unspecified} \end{array} \tag{33}$$

We also have that $\phi^*(t=T) = \phi^*(\tau=0) = 0$, and hence by (33) and the transversality condition (5) it follows that $p_3^o = 0$. It should be noted that p_2^o remains unspecified on the terminal surface. It will be shown that p_2^o is determined by conditions for extremals to reach C_4. Recalling the extremal control at $t = T$ and (13), we also must have that

$$a_1 p/a_2 < (-p_2^o). \tag{34}$$

Since $v(\tau=0) < 0$ and $\frac{dv}{d\tau} > 0$ by (12), there will be a switch in tactics in "backwards time" at $\tau = \tau_1(C_4) = \tau_1$. (In this section the notation τ_1 refers to extremals leading to C_4 when it is not denoted otherwise.) Thus, the extremal control is given by

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_2 \\ 0 & \text{for } t_2 \leq t \leq T, \end{cases} \tag{35}$$

where $t_2 = T - \tau_1$ and t_2 is such that

$$2b_1x_1(t_2)x_2^o + b_2(x_2^o)^2 = a_2y^2(t_2). \tag{36}$$

From a backwards integration of the adjoint equations and the maximum principle, the time of the switch in tactics is given by

$$w = \cosh\sqrt{a_2b_2}\ \tau_1 = \frac{a_1}{p_2^o}\frac{(b_1p_2^o + b_2p)}{(a_1b_1 - a_2b_2)} = w(p_2^o). \tag{37}$$

We observe that from what region of the initial state space extremals lead to C_4 is determined by the value assigned to p_2^o. In other words, the value assigned to p_2^o determines what initial force levels x_2^o are to be totally destroyed. It seems reasonable to argue that $(-p_2^o)$ cannot exceed q, since otherwise we would implicitly be valuing X_2 survivors greater than stated for the problem. A more precise argument, of course, is to compare payoffs (i.e. P_4 and P_5, as done in section 5.1) for extremals leading to C_4 and C_5 from the same initial point P^o corresponding to $p_2^o > q$, but this has not proven to be computationally tractable. Thus, recalling (34), we must have

$$a_1p/a_2 < (-p_2^o) \leq q. \tag{38}$$

As done previously, we combine the extremal control (35) with (14) to obtain

$$(b_1x_1(t_2))^2 - 2(R-1)b_2x_2^ob_1x_1(t_2) + a_1b_1y_o{}^2 - s^2 - (R-1)(b_2x_2^o)^2 = 0, \qquad (39)$$

which may be solved for $x_1(t_2)$ to yield

$$b_1x_1(t_2) = b_2x_2^o(R-1) + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_o{}^2}, \qquad (40)$$

whence the result shown in Table I by requiring that the quantity under the radical sign is non-negative. The other entries for C_4 in Table I correspond to other parts of the terminal surface being impossible to reach.

It remains to show that optimal paths can only lead to C_4 from P^o when

$$a_1b_1y_o{}^2 \leq s^2 + A(b_2x_2^o)^2, \qquad (41)$$

where $A = A(R,z)$ and $z = \cosh\sqrt{a_2b_2}\ \tau_1(C_5)$. A backwards integration of the state euqations (1) for $0 \leq \tau \leq \tau_1(C_4) = \tau_1$ with $x_1(\tau=0) = x_1(t_2)$ and $x_2(\tau=0) = 0$ yields that

$$x_1(t_2) = \frac{b_2x_2^o}{b_1(w-1)}, \qquad (42)$$

since $x_2(\tau=\tau_1) = x_2^o$ and $x_1(\tau=\tau_1) = x_1(t_2)$. Combining (39) and (42), we obtain

$$s^2 + A(R,w)(b_2x_2^o)^2 - a_1b_1y_o{}^2 = 0. \qquad (43)$$

For constant R, A may be considered to be a function of w. Considering the first two derivatives of $A(w)$ and $\lim_{w\to\infty} A(w)$, it is readily seen that for $1 \leq w \leq +\infty$, $A(w)$ has a global maximum at $w^* = R/(R-1)$ with $A(w^*) = R(R-1) > 0$. This latter result implies that for P^o satisfying (41) the quantity under the radical sign in (40) is non-negative (i.e. the corresponding entry in Table I is satisfied). It should be observed that $w(-p_2^o)$ is a strictly increasing function of $(-p_2^o)$ with

$$w\left(-p_2^o = \frac{a_1 p}{a_2}\right) = 1 \quad \text{and} \quad w(-p_2^o = q) = z = \frac{R-\delta}{R-1} \leq \frac{R}{R-1}. \tag{44}$$

Since for $1 \leq w \leq w^*$, $A(w)$ is a strictly increasing function; then for $\frac{a_1 p}{a_2} \leq (-p_2^o) \leq q$, $A(w(-p_2^o))$ is a strictly increasing function whence (41). Also, combination of (37), (40) and (42) yields that

$$(-p_2^o(P^o)) = \frac{a_1 p\{b_2 x_2^o(R-1) + \sqrt{S(P^o,R)}\}}{a_2\sqrt{S(P^o,R)}} \tag{45}$$

where

$$S(P^o,R) = s^2 + R(R-1)(b_2 x_2^o)^2 - a_1 b_1 y_o{}^2. \tag{46}$$

Equation (45) shows the dependence of dual variable values on the terminal surface upon the initial force levels P^o of an extremal. It seems appropriate to observe that for $a_1 p/a_2 \leq (-p_2^o(P^o)) \leq q$ we have that $0 \leq \tau_1(C_4) \leq \tau_1(C_5)$ and also $\tau_1(C_4) = \tau_1(P^o)$. The value implicitly assigned to X_2 survivors through p_2^o is adjusted so that the X_2 forces are just annihilated at $t = T$. Let us also note that

by construction we have made $v(\tau=\tau_1(P^o)) = 0$ in our determination of $(-p_2^o(P^o))$. The above should make clear the reason that p_2^o must be unspecified on C_4: for extremals corresponding to a given value of $x_1(T)$ and leading to C_4 there are an infinite number of different values for $\tau_1(P^o)$ for these extremals that result in this $x_1(T)$.

5.3. Determination of Dispersal Surfaces.

It seems appropriate to note here some relationships between the inequalities which define the domains of controllability in Table I. It is always true that

$$s^2 + B(b_2x_2^o)^2 \leq s^2 + (R-1)(b_2x_2^o)^2. \qquad (47)$$

For $R - \sqrt{R(R-1)} \leq \delta < 1$, the following hold

$$s^2 + A(b_2x_2^o)^2 \leq s^2 + (R-1)(b_2x_2^o)^2, \qquad (48)$$

$$Rs^2\{1 - 1/z^2\} \leq s^2 + (R-1)(b_2x_2^o)^2, \qquad (49)$$

$$s^2 + A(b_2x_2^o)^2 \leq s^2 + B(b_2x_2^o)^2, \qquad (50)$$

$$Rs^2\{1 - 1/z^2\} \leq s^2 + B(b_2x_2^o)^2. \qquad (51)$$

In (51), there is strict inequality for $x_1^o > 0$ and $R - \sqrt{R(R-1)} < \delta < 1$. For $0 \leq \delta < R - \sqrt{R(R-1)}$, we also have

$$s^2 + A(b_2x_2^o)^2 > s^2 + B(b_2x_2^o)^2, \qquad (52)$$

$$Rs^2\{1 - 1/z^2\} > s^2 + B(b_2x_2^o)^2. \qquad (53)$$

The proofs of the above are straight-forward and omitted.

When $0 \leq \delta < R - \sqrt{R(R-1)}$, the above inequalities (52) and (53) imply that $D(C_1)$ and $D(C_5)$ for extremals on which $\varphi^*(t) = 0$ for $0 \leq t \leq T$ overlap (i.e. there is a region in the initial state space from which extremals lead to both C_1 and C_5) and similarly for $D(C_1)$ and $D(C_4)$ when $a_1b_1{y_o}^2 \leq s^2 + A(b_2x_2^o)^2$ in $D(C_4)$. Hence, we must consider step (e) of section 4.'s solution procedure.

First, we consider the determination of the dispersal surface which separates regions of the initial state space from which optimal paths lead to C_1 and C_5. Let us observe that for $z^2 > R/(R-1)$ points exist for which

$$s^2 + B(b_2x_2^o)^2 \leq a_1b_1{y_o}^2 \leq Rs^2\{1 - 1/z^2\}. \qquad (54)$$

Setting $a_1b_1{y_o}^2 = s^2 + B(b_2x_2^o)^2$ in the expressions (26) and (28) for P_5 and P_1 respectively, it is easily shown that it is optimal to go to C_5 from such an initial point. The dispersal surface is the locus of points in the initial state space for which P_1 equals P_5 and extremals lead to both C_1 and C_5 on both sides of this surface. Equating (26) and (28), the locus of points for which $P_1 = P_5$ is given by

$$a_1b_1{y_o}^2 = Rs^2 - R\{b_1x_1^o[z^2(R-1) + R]/(2R) + b_2x_2^o\}^2/z^2. \qquad (55)$$

Furthermore, a dispersal surface actually exists, since it is readily shown that

$$Rs^2 - R\{b_1x_1^o[z^2(R-1) + R]/(2R) + b_2x_2^o\}^2/z^2 < Rs^2\{1 - 1/z^2\}, \quad (56)$$

when $z^2 > R/(R-1)$. (56) and an above result mean that on both sides of the candidate dispersal surface extremals lead to both C_1 and C_5. Observing that $\frac{\partial P_1}{\partial y_o} \geq \frac{\partial P_5}{\partial y_0} \geq 0$, it is readily seen that $P_1 \geq P_5$ for all P^o such that $a_1b_1y_o \geq Rs^2 - R\{b_1x_1[z^2(R-1) + R]/(2R) + b_2x_2^o\}^2/z^2$. Since we also must have $a_1b_1y_o{}^2 < R\{s^2 - (b_1x_1^o)^2\}$ for extremals leading to C_5, the above dispersal surface only exists for $0 \leq x_1^o < 2Rb_2x_2^o/\{b_1[z^2 - R(z-1)^2]\}$.

Based on the above results there may also be a dispersal surface which separates optimal paths leading to C_1 and C_4 for $0 \leq x_2^o \leq b_1x_1^o[z^2 - R(z-1)^2]/(2Rb_2)$. Its determination is similar to the above, although the details are messier and haven't been completely worked out. Let

$$a_1b_1y_o = g(P^o,R,z), \quad (57)$$

be the locus of points for which $P_1 = P_4$. Then, the boundary, which separates optimal paths leading to C_1 and C_4, is given by

$$a_1b_1y_o = h(P^o,R,z), \quad (58)$$

where

$$h(P^o,R,z) = \begin{cases} s^2 + B(b_2x_2^o)^2 & \text{for } g(P^o,R,z) < s^2 + B(b_2x_2^o)^2 \\ g(P^o,R,z) & \text{for } s^2 + B(b_2x_2^o)^2 \leq g(P^o,R,z) \leq s^2 + A(b_2x_2^o)^2 \\ s^2 + A(b_2x_2^o)^2 & \text{for } g(P^o,R,z) > s^2 + A(b_2x_2^o)^2. \end{cases} \quad (59)$$

It should be noted that this boundary surface is a dispersal surface only when extremals to both C_1 and C_4 lie on both sides of $a_1b_1y_o^2 = g(P^o,R,z)$.

6. Structure of the Optimal Allocation Policies.

For square-law attrition it may be shown that the allocation of fraction of fire is always 0 or 1. In section 5 we have discussed the fact that the coefficient of the control variable φ in the Hamiltonian $H(t, x_i, p_i, \varphi)$ cannot be equal to zero over any finite interval of time. This excludes the possibility of a singular control [15], and maximization of the Hamiltonian, which is a linear function of φ, yields that all fire is concentrated on one target type. This is not surprising, since our model assumes complete and instantaneous information [2] and that fire may be immediately shifted to a new target once the old one has been destroyed [5], [18].

With reference to Tables I and II, the condition that $a_1 b_1 > a_2 b_2$ may be interpreted to mean that there is more long range return for Y to engage X_1, i.e. more Y's will survive if this is done. Hence, when Y wins, he always engages X_1's while they are available. The condition $a_1 p < a_2 q$ means that at the end of battle there is greater payoff per unit time per Y soldier to engage X_2 not considering X_1's greater attrition effect against Y (short term gain at end of battle).

By the maximum principle and the well-known interpretation of the dual variables [1], Y always allocates his fire entirely to the target type yielding the greatest marginal return. However, marginal return evolves differently in winning or losing causes. When Y loses, he may switch from firing at X_1 entirely to firing at X_2 entirely

to firing at X_2 entirely before the X_1 force has been annihilated. This happens when Y assigns utility to survivors of force type X_2 in excess of their kill rate against Y as compared to force type X_1, and X_1 is abundant enough not to be destroyed before the battle ends. Under these circumstances, whenever the X_2 forces are annihilated, this occurs at the very end of battle.

In this way, we see that tactics may depend on force levels. Moreover, the time at which the ranking of target priorities switches itself may depend upon force levels. We also see that Y's target priorities only switch with time in a losing case. This has occurred since a boundary condition at $t = T$ on one of the dual variables is dependent upon values of the state variables by a transversality condition. It may be shown that the structure of optimal allocation policies is different for the prescribed duration battle.

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Appendix B. Lanchester-Type Models of Warfare and Optimal Control.

1. Introduction.

The optimization of the dynamics of combat is studied through a sequence of idealized models by use of mathematical theory of optimal control. The models are for combat over a period of time described by Lanchester-type equations with a choice of tactics available to one side and subject to change with time. A sequence of models is examined for each of two types of choice situation:

selection of target type for engagement,

regulation of firing rate.

Optimal tactics are discussed with reference to the influence of combatant objectives, termination conditions of conflict, type of attrition process, variable attrition rates, and limited ammunition. Implications for intelligence, command and control systems, and human decision making are pointed out. The use of such optimal control models for guiding extensions to differential games is discussed.

In this appendix we examine the structure of optimal allocation policies for tactical situations describable by Lanchester-type equations of warfare. We hope to provide insight into such questions as

(1) How should targets be selected?

(2) Do target priorities change with time?

(3) Does the number of target types effect the selection?

(4) Do battle termination circumstances effect the optional allocation policies?

(5) How does the nature of the attrition process effect target selection?

(6) What is the effect of ammunition constraints?

(7) How does the uncertainty and confusion of combat effect the optimal selection rules?

Our theory of tactical allocation is developed through the examination of a sequence of simplified models. These combat models are too simple to be taken literally but should be interpreted as indicating general principles to serve as hypotheses for subsequent computer simulation studies or field experimentation.

In 1964 Dolansky [10] noted that the Lanchester theory of combat was insufficiently developed in the area of target selection for combat between heterogeneous forces (optimal control/differential games). Even the two references cited by him, Weiss [32] and Isbell and Marlow [16], have been subsequently extended by this author [26], [28]. Since Dolansky's article, no further examples have been published in the literature except for the ones in Isaacs' book [15]. This previous work had never systematically investigated the dependence of tactics upon model form.

We examine several idealized combat situations described by Lanchester-type equations over a period of time with choices of tactics available to one side and subject to change with time. We consider models for two types of choice situations

(1) selection of target type,

(2) regulation of firing rate.

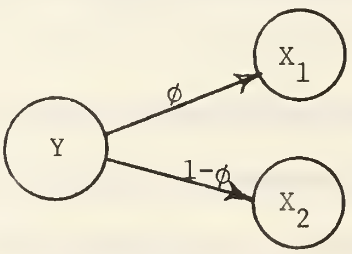
These problems are solved by the mathematical theory of optimal control. A further elaboration on solution development is to be found in our report [25].

In the first sequence of models we examine the effect on optimal target selection of the following factors: objectives of the combatants, termination conditions of the conflict, number of target types, some special cases of time dependent attrition rates, and type of attrition process. We then examine a sequence of models to see how ammunition limitations effect firing rates. Next we discuss two-sided extensions of such problems but point out the value of studying one-

sided problems as considered in this appendix. Finally, various implications of the models are discussed.

2. Target Selection.

We begin by considering the simplest situation of target selection: combat between an X-force of two force types (for example, riflemen and grenadiers) and a homogeneous Y-force (for example, riflemen only). This situation is shown diagrammatically below.



It is the objective of the Y-force commander to maximize his survivors at the end of battle at time T and minimize those of his opponent (considering weighting factors p, q and r). This is accomplished through his choice of the fraction of fire, ϕ, directed at X_1. We may study this idealized tactical allocation problem in two types of scenarios: (1) a battle lasting a specified time, T or (2) a battle lasting until one side or the other is totally annihilated.

2.1. Battle of Prescribed Duration.

Mathematically the problem may be stated as

$$\underset{\phi(t)}{\text{maximize}}\ ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where

$p, q,$ and r are weighting factors assigned to surviving forces,

x_1, x_2 and y are average force strengths,

a_1, a_2, b_1 and b_2 are constant attrition rates,

and ϕ is fraction of Y-fire directed at X_1.

This problem may be solved by routine application of Pontryagin maximum principle* [22]. The solution is shown in Table I. In this present analysis we have not considered those subcases when a state variable is reduced to zero. It may be shown that the structure of the optimal allocation policies is not fundamentally altered by such an occurrence.

With reference to Table I, we see that two characteristics of the optimal allocation policies for this particular prescribed duration battle are:

(1) concentration of all fire on one target type,

(2) independence of allocation from force levels.

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*We employ the equivalent version commonly used in this country (see p. 108 of [9]).

TABLE I. Solution to Target Selection Problem

Battle of Prescribed Duration with Constant Attrition Rates

Nonrestrictive assumption: $a_1 b_1 > a_2 b_2$

	Optimal Control
Case A: $a_1 p \geq a_2 q$	$\phi^*(t) = 1$ for $0 \leq t \leq T$
Case B: $a_1 p < a_2 q$	(a) for $\tau_1 > T$ $\phi^*(t) = 0$ for $0 \leq t \leq T$ (b) for $\tau_1 < T$ $\phi^*(t) = 1$ for $0 \leq t \leq T - \tau_1$ $\phi^*(t) = 0$ for $T - \tau_1 \leq t \leq T$

Note: τ_1 is determined from the transcendental equation

$$r\sqrt{\frac{b_2}{a_2}} \sinh \sqrt{a_2 b_2}\, \tau_1 + q \cosh \sqrt{a_2 b_2}\, \tau_1 = \frac{a_1(b_1 q - b_2 p)}{(a_1 b_1 - a_2 b_2)}$$

We shall later see that when there are more than two target types in this scenario, the solution possesses these same characteristics (even when the attrition rates change over time). Both these characteristics, however, are consequences of the assumed model form.

The first characteristic, concentration of effort on one alternative, is a consequence of the "square-law" attrition process for the X-forces. (We shall refer to attrition as being a "square-law" process when the casualty rate is proportional to the number of enemy firers and as being a "linear-law" process when it is proportional to the product of the number of enemy firers and remaining targets.) It may be shown that this makes the existence of a singular control [18] impossible, and hence the optimal allocation policies are extreme points in the control variable space.

There is, however, a very simple principle which underlies the above mathematical formalities: concentration of effort when constant marginal returns are obtained from the alternatives and the total effort is limited. Constant marginal effect over time per unit of weapon system is a property of the "square-law" attrition process, for let us consider the X_1-force attrition (when $\phi = 1$)

$$\frac{\left(-\frac{dx_1}{dt}\right)}{y} = a_1 = \left\{\begin{matrix}\text{rate of casualties produced per} \\ \text{unit of Y-force weapon system}\end{matrix}\right\}$$

Thus there is a constant (or non-diminishing) marginal effect over time. This should be contrasted with the situation for a "linear-law" attrition of the X_1-forces

$$\frac{\left(-\dfrac{dx_1}{dt}\right)}{y} = a_1 x_1 = \begin{Bmatrix}\text{rate of casualties produced per}\\ \text{unit of Y-force weapon system}\end{Bmatrix}$$

In this case we have diminishing effects over time from allocating a unit of Y-force weapon system against X_1, and a division of total effort (i.e., fraction of fire) may be called for. B. Koopman's 1953 article [21] contains an excellent discussion of such principles which underlie such an optimization problem. Presently, we shall verify these heuristic arguments in a mathematically precise fashion when we consider a dynamic model, which considers the interaction of forces over time, in which both X-force target types undergo "linear-law" attrition. This fundamental difference in the structure of optimal allocation policies based on the nature of target attrition makes the determination of the appropriate attrition process an essential task of analysis.

The second characteristic, target selection independent of force levels, is due to the combination of the "square-law" attrition process for the X-force types and the fixed battle length, T. It is seen that for the battle of prescribed duration target selection depends only on the attrition rates of the various force types and relative weights assigned to surviving force types. This is not surprising, since it is easily seen that the adjoint differential equations are independent of the state variables, and the values of the dual variables at the end of battle $t = T$ are independent of force strengths. It is recalled that a dual variable represents the rate of change of the payoff (battle outcome as measured by the value of surviving forces

at $t = T$) with respect to a particular state variable [3]. Thus, if if $V = ry(T) - px_1(T) - qx_2(T)$, then $p_1(t) = \frac{\partial V}{\partial x_1}(t)$, etc. Hence, the boundary conditions are given for the dual variables at the end of the battle $t = T$ as $p_1(t{=}T) = \frac{\partial V}{\partial x_1}(t{=}T) = -p, p_2(t{=}T) = -q, p_3(t{=}T) = r$.

It seems appropriate to discuss further the interpretation of the solution shown in Table I. From the above definition of the dual variables,

$$a_1 p_1(t) = \begin{pmatrix}\text{effect on outcome per} \\ \text{unit time for engaging } X_1\end{pmatrix} = \begin{pmatrix}\text{kill rate of} \\ Y \text{ against } X_1\end{pmatrix} \times \begin{pmatrix}\text{effect on outcome per} \\ \text{unit of } X_1 \text{ destroyed}\end{pmatrix}.$$

Hence, the condition $a_1 p < a_2 q$ means that at the end of the battle (recall that $p_1(t{=}T) = -p$, etc.) there is greater effect on battle outcome (as measured by value of survivors) per unit time per soldier for Y to engage X_2 (short term gain at the end of battle). The value of the dual variable, for example, $p_1(t)$ reflects both the value assigned X_1-force survivors and the dynamic interaction of forces over time through the Lanchester-type equations. Hence, it also accounts for the effectiveness of X_1 against Y. The quantity $a_1 b_1$ may be interpreted as representing the instantaneous rate of destruction of the X_1-force kill rate against the Y-force per unit of Y-force. Then $a_1 b_1 > a_2 b_2$ means that there is greater strategic value for engaging the X_1-force, i.e., more long range return. Thus, case A of Table I corresponds to where there is both more long range

and also short range return for engaging X_1. Case B corresponds to more short term gain at the end of the battle for engaging X_2, but more long range return for engaging X_1. It is easily shown that case A results when Y values surviving X-forces in direct proportion to their kill rate against the Y-force, i.e., $p/q = b_1/b_2$. A switch in tactics (target priority) is seen to occur for this model only when value is not assigned to survivors of a target-type in proportion to their destructive capability (kill rate).

The maximum principle may be interpreted as saying that a target type from several alternatives is engaged when such an engagement yields the greatest favorable effect on battle outcome per unit time. turns out, though, that the evolution of target engagement return is dependent upon the scenario chosen for the study of the problem. This is clearly seen when we examine the "fight to the finish." This is a special case of a terminal control battle (the combat ends only when the course of battle has been steered to a prescribed end state) and is chosen for mathematical convenience.

2.2. Terminal Control Battle.

We consider the similar problem of

$$\max_{\phi(t)} \; ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ unspecified}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2$$

$x_1, x_2, y \geq 0$, $0 \leq \phi \leq 1$, and with terminal states defined by (1) $x_1(T) = x_2(T) = 0$ and (2) $y(T) = 0$. The terminal surface of this problem is seen to consist of five parts:

C_1: $x_1(T) = 0$, $x_2(T) > 0$, $x_3(T) = 0$

C_2: $x_1(T) = 0$ before $x_2(T) = 0$, $x_3(T) > 0$

C_3: $x_1(T) = 0$ after $x_2(T) = 0$, $x_3(T) > 0$

C_4: $x_1(T) > 0$, $x_2(T) = 0$, $x_3(T) = 0$

C_5: $x_1(T) > 0$, $x_2(T) > 0$, $x_3(T) = 0$

The above problem was first studied by Isbell and Marlow [16], but a complete solution was first obtained by us in a previous paper [26]. We have also extended the solution principles to terminal control differential games, for which we have outlined a general solution procedure and have used it to solve the supporting weapon system game of H. K. Weiss [28]. The solution to this problem appears complex (see Table I of [26]) but may be described in a particularly simple fashion (for the nonrestrictive assumption that $a_1 b_1 > a_2 b_2$).

In contrast to the battle of prescribed duration, the optimal target engagement may depend on initial force levels. When Y wins, he engages X_1 until depletion before X_2. When Y loses, he may switch from firing at X_1 entirely to firing at X_2 entirely before the X_1 force has been annihilated. This happens when survivors of

force-type X_2 are assigned utility in excess of their kill rate as compared with force-type X_1, and certain relationships hold between initial force strengths. This dependence of the optimal allocation on initial strengths has been caused by the fact that values of dual variables at $t = T$ are dependent upon values of the state variables. This happens in terminal control attrition problems where a value of a state variable is specified at the terminal surface (and hence the value of the corresponding dual variable is unspecified but may be determined from the transversality condition $H(t = T,x,p,\phi) = 0$, where $H(t,x,p,\phi)$ denotes the Hamiltonian).

2.3. Prescribed Duration Battle with Several Target Types.

We have considered the first two problems in order to contrast the effect of the battle termination conditions upon the structure of the optimal allocation policies. Another factor that we can examine is the number of target types. For the prescribed duration battle, certain facets which tended to be obscured in the scenario with two target types are brought into sharp focus. In a companion paper [30] we present the supporting analysis details, and in this section we summarize solution properties in order that they may be contrasted with those of other scenarios in this first sequence of target selection problems.

We consider the following prescribed duration problem:

$$\underset{\phi_i(t)}{\text{maximize}}\ vy(T) - \sum_{i=1}^{n} w_i x_i(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = -\phi_i a_i y \quad \text{for } i = 1,\ldots,n$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i x_i$$

$$x_i, y \geq 0, \quad \phi_i \geq 0, \quad \text{and} \quad \sum_{i=1}^{n} \phi_i = 1,$$

where all symbols are used in the same sense as previously. The solution to this problem turns out to be a generalization of that in section 2.1. However, certain aspects receive greater emphasis to provide us with a deeper understanding of the phenomena under study. In particular, we considered two subcases, denoted as case A and case B, in the solution to the previous problem of section 2.1. (see Table I). When there are several target types, the generalization of the subcases which we must distinguish is as follows:

Case A, enemy survivors valued in direct proportion to their kill rate against Y-force,

Case B, enemy survivors <u>not</u> valued in direct proportion to their kill rate against Y-force.

In the first instance, case A, it may be shown that target priorities keep their same relative ranking over time. Assuming for the moment that no force type is totally annihilated during the course of battle, the optimal allocation policy is

$$\phi_i^*(t) = \delta_{ij} \quad \text{for} \quad 0 \le t \le T,$$

where δ_{ij} is the Kronecker delta and is equal to 1 if $i = j$ and zero otherwise, and j is the index such that $a_j b_j = \max(a_1 b_1, \ldots, a_n b_n)$. If the highest priority target type is exterminated during such a battle, then fire is merely shifted to the next highest priority target. Hence, when one values enemy survivors in proportion to their kill rate against you, i.e., $w_i = kb_i$ for $i = 1, \ldots, n$, the optimal tactic is to concentrate all fire on a single target type until it is entirely destroyed. The sole criterion for target selection in this instance is the quantity $a_i b_i$, which may be interpreted to be the rate of destruction of enemy attrition capability for his i^{th} force type (see section 2.1.).

In case B, it may be shown that there will be at least one switch of target priorities if the battle lasts long enough. Again assuming that no force type is totally annihilated during the course of battle, it may be shown that for $\tau_1 \ge T$ the optimal allocation policy is given by

$$\phi_i^*(t) = \delta_{in} \quad \text{for} \quad 0 \le t \le T,$$

where j is the index such that $a_j w_j = a_j[-p_j(t=T)] = \max(a_1 w_1, \ldots, a_n w_n)$ and we have arranged our indexing so that this is the last index, i.e., $j = n$, $p_j(t)$ is the dual variable corresponding to the state variable x_j, and τ_1, the "backwards" time $\tau = T - t$ (measured from the end of battle) of the first switch in target selection, will be given below. The battle must last longer than τ_1 for a switch in target priorities

to occur. Assuming that it does, we now state what conditions are necessary for a change in target selection and the "backwards" time at which the change occurs, τ_1.

In developing the solution to this problem we work backwards from the end of battle, $t = T$. Let k be the index of the target type to which fire is first shifted in "backwards" time, τ. Then it may be shown that necessary conditions for fire to be switched to the k^{th} target type are that $a_k b_k > a_n b_n$ and $\frac{b_k}{b_n} > \frac{w_k}{w_n}$, i.e., we shift fire to a target type which causes attrition in a greater proportion than the ratio of values placed upon survivor from the target type which yields the greatest direct return at the end of battle. The target type to which fire is shifted has index k determined by

$$R_k = \min_{\substack{R_i>0 \\ a_i b_i > a_n b_n}} (R_1, \ldots, R_{n-1}),$$

where

$$R_i = \frac{a_i(b_i w_n - b_n w_i)}{a_i b_i - a_n b_n}, \quad \text{for} \quad i = 1, \ldots, n-1.$$

The "backwards" time of switch of fire to the k^{th} target, τ_1, is determined from the transcendental equation

$$w_n \cosh\sqrt{a_n b_n}\, \tau_1 + v\sqrt{\frac{b_n}{a_n}} \sinh\sqrt{a_n b_n}\, \tau_1 = \frac{a_k(b_k w_n - b_n w_k)}{a_k b_k - a_n b_n}$$

This is seen to be a generalization from the problem with two X-force target types.

Barring the extinction of a force type, the optimal allocation policy for $\tau_1 < T$ but $\tau_2 \geq T$ is given by

$$\phi_i^*(t) = \delta_{ik} \quad \text{for} \quad 0 \leq t \leq T - \tau_1$$

$$\phi_i^*(t) = \delta_{in} \quad \text{for} \quad T - \tau_1 \leq t \leq T,$$

where τ_1 is determined by (1) and a similar expression exists for τ_2, the "backwards" time of the second change. A similar solution with two changes in the optimal allocation policy exists when $\tau_2 \leq T$ but $\tau_3 > T$. The number of switches in target engagement depends on the length of battle, T, and as the battle progresses forward in time fire is always shifted to a target type for which both $a_i b_i$ and $\frac{b_i}{w_i}$ are smaller. If the battle lasts long enough, then during the initial stages all fire is concentrated on the X-force type for which both the quantities $a_i b_i$ and $\frac{b_i}{w_i}$ are larger than any other.

Although the details differ, the optimal allocation policies are seen to have the same structure as that for just two target types (see section 2.1.):

(1) concentration of all fire on one target type,

(2) independence of allocation from force levels.

The addition of more target types has not changed the nature of the problem. This problem's explicit solution is a generalization of that with two X-force target types.

It is of interest to ask whether the optimal tactic will always be to concentrate fire on only one target type (bang-bang optimal control). The answer to this question turns out to be "no" as

consideration of a "linear-law" attrition process for the X-force target types will show in section 2.5. An extensive heuristic discussion of the principles involved has been given in section 2.1.

2.4. <u>Some Special Cases of Time Dependent Attrition Rates</u>.

In the previous idealizations of combat that we have considered above, we have assumed that all the Lanchester attrition-rate coefficients were constants. In reality, this coefficient depends on numerous factors some of which are as follows: hit probabilities, weapon system projectile-target lethality characteristics, rates of fire, rate of target acquisition. These factors themselves may be range dependent or change over time. S. Bonder [6], [7] has developed explicit formulas for relating the Lanchester attrition-rate coefficient to weapon system performance characteristics such as those mentioned above.

Thus, it seems appropriate to examine idealized combat situations in which the attrition rates are time dependent. Solutions have been obtained to variable-coefficient Lanchester-type equations for a square-law attrition process between two homogeneous forces under very general circumstances [29]. However, such solutions are, in most instances, so complicated that they cannot be applied to corresponding optimal control problems to yield easily interpretable analytic results which provide insight into the structure of the optimal allocation policies. There is a class of variable coefficient Lanchester-type equations (combat between two homogeneous forces when the attrition rates are

variable provided that their quotient is a constant) which possess a solution no more complicated than the solution to the constant coefficient case [27]. Hence, when one considers optimal control problems for Lanchester combat between forces with time dependent attrition rates, there are some special instances of practical interest (reflecting the physical situation in which two weapon systems cause attrition in a proportional fashion at all times) that are not much more mathematically complicated than the idealized situations we have considered in sections 2.1 and 2.3 above. Again, we shall merely summarize results, leaving the details for a companion paper [30].

We consider the following prescribed duration battle:

$$\max_{\phi(t)} \text{imize } ry(T) - px_1(T) - qx_2(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1(t)y$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2(t)y$$

$$\frac{dy}{dt} = -b_1(t)x_1 - b_2(t)x_2$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1.$$

We assume that both X-force weapon systems are such that

$b_1(t) = k_{b_1}h(t)$ and $b_2(t) = k_{b_2}h(t)$.

In the special case when the Y-force values surviving X-force types in direct proportion to their kill rate against the

Y-force, i.e., $p/q = k_{b_1}/k_{b_2} = b_1(t)/b_2(t) = b_1(t=T)/b_2(t=T)$, the optimal control law takes a particularly simple form

$$\phi^*(t) = \begin{cases} 1 & \text{for } a_1(t)b_1(t) > a_2(t)b_2(t) \\ 0 & \text{for } a_1(t)b_1(t) < a_2(t)b_2(t) \end{cases}$$

In this instance target selection depends only on the product of attrition rates which may be interpreted as the rate of destruction of enemy kill rate. All fire is concentrated on one of the target types depending on which target type has the larger attrition-rate product. Target priority is subject to change over time as the ranking of the target types on this decision criterion changes. It is conceivable that the optimal tactic may be to shift fire from one target type to the other several times over the course of battle with the duration of battle not having any effect. Observe that no assumptions at all have been made on the Y-force attrition rates against X_1 and X_2, i.e., $a_1(t)$ and $a_2(t)$.

At this point we further assume that $a_1(t) = k_{a_1}h(t)$ and $a_2(t) = k_{a_2}h(t)$. This means that not only is the ratio of the X-force weapon system attrition rates against the Y-force constant but also the ratio of the Y-force effectiveness against each of the two X-force types. Furthermore, all four attrition rates have the same time dependence except for constant factors. The solution is shown in Table II. In this special case (under the assumptions noted above), we see that the structure of the optimal allocation policies when the attrition rates are variable is essentially identical to that when

TABLE II. Solution to Target Selection Problem

Battle of Prescribed Duration with Variable Attrition Rates

Special assumption: $a_1(t)/a_2(t) = k_{a_1}/k_{a_2}$, $b_1(t)/b_2(t) = k_{b_1}/k_{b_2}$, and $a_1(t)/b_1(t) = k_{a_1}/k_{b_1}$

Nonrestrictive assumption: $k_{a_1}k_{b_1} > k_{a_2}k_{b_2}$

	Optimal Control
Case A: $k_{a_1}p \geq k_{a_2}q$	$\phi^*(t) = 1$ for $0 \leq t \leq T$
Case B: $k_{a_1}p < k_{a_2}q$	(a) for $\tau_1 > T$ $\phi^*(t) = 0$ for $0 \leq t \leq T$ (b) for $\tau_1 < T$ $\phi^*(t) = 1$ for $0 \leq t \leq T - \tau_1$ $\phi^*(t) = 0$ for $T - \tau_1 \leq t \leq T$

Note: τ_1 is determined from the transcendental equation

$$r\sqrt{\frac{k_{b_2}}{k_{a_2}}}\sinh\{\sqrt{k_{a_2}k_{b_2}}\int_0^{\tau_1} h(\tau)d\tau\} + q\cosh\{\sqrt{k_{a_2}k_{b_2}}\int_0^{\tau_1} h(\tau)d\tau\} = \frac{k_{a_1}(k_{b_1}q - k_{b_2}p)}{(k_{a_1}k_{b_1} - k_{a_2}k_{b_2})}$$

they are constant. Only the time scale has been transformed (in [27] we first made this type of observation).

The extensions to more than two target types in each of the two cases considered above possess essentially the same solution details as those noted here. These extensions are similar to those we considered in section 2.3.

2.5. Selection of Targets Undergoing "Linear-Law" Attrition.

So far the state equations have described combat according to the Lanchester square law in which attrition of a target type is proportional to the number of each force type firing at it. Weiss [31] has given a thorough discussion of the conditions which lead to this. These conditions include that "each unit is informed about the location of the remaining opposing units so that when a target is destroyed, fire may be immediately shifted to a new target." It is noted that the control theory models which we have considered so far have implicitly assumed perfect information.

Another model for attrition is the Lanchester linear law in which the average decrease of a target type is proportional to the product of the average number of targets remaining and the number of each force type firing at it. Such a dependence can arise under two general circumstances: (1) fire is uniformly distributed over a constant target area ("area fire") or (2) the mean time of target acquisition is much larger than target destruction time and is inversely proportional to target density. The first circumstance corresponds to the simplest case of partial information. Again quoting Weiss [31],

we assume that units are informed about the general areas in which opposing units are located, but are not informed about the consequences of their own fire. Thus, we see that we may account for some changes in the information set by modifying the description of combat. Brackney [8] has shown that "aimed fire" may lead to linear-law attrition when target acquisition times are considered, and are as postulated above.

Thus, we consider the following problem in which the X-forces' attrition obeys a linear-law process and the Y-forces' attrition obeys a square-law process:

$$\underset{\phi(t)}{\text{minimize}}\ ry(T) - px_1(T) - qx_2(T) \quad \text{with} \quad T \quad \text{specified}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 x_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1.$$

In a future paper* we shall present the analysis details upon which our present summary is based (see also pp 91-105 of [25]). Since the state and adjoint equations do not readily yield an analytic solution, we have not been able to obtain explicit expressions for certain model parameters. However, we can still discuss all the qualitative characteristics of the structure of the optimal allocation policies.

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*See Appendix D.

There is a fundamental difference between the solution to this problem and those considered previously: the optimal allocation, ϕ^*, may be other than 0 or 1. In contrast to the previous problems, the optimal allocation policy does not have to be an extreme point of the control variable space at all times. We may have a singular solution [18] for which the necessary condition of maximizing the Hamiltonian (with respect to the control variable) does not provide us with a well-defined expression for the extremal control. We shall call the part of an optimal trajectory on which the maximum principle cannot be used to determine the control a singular subarc. Then the term "singular solution" will be used to refer to any optimal trajectory which contains one or more singular subarcs.

Singular solutions can only occur when the Hamiltonian (denoted as H) is a linear function of the control variables. When this is so, then if $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time (or, another way to say this, the coefficient of ϕ vanishes identically for a finite interval of time), the maximum principle does not determine the control. Observe that when $\frac{\partial H}{\partial \phi} = 0$ and H is a linear function of ϕ, <u>all</u> feasible values of ϕ are optimal. All problems, however, for which the Hamiltonian is a linear function of the control variables do not have singular subarcs in their solution. For example, it may be shown that such a singular control is impossible for the problems considered above in sections 2.1 through 2.4, since it is impossible for $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ for a finite interval of time when $\frac{\partial H}{\partial \phi} = 0$.

For the problem at hand, it may be shown that a necessary condition for a singular subarc to yield the maximum return [20] is satisfied.

The optimal battle trajectories are constructed by working backwards from all possible end points of this idealized battle. Consideration is given to both the optimal control at the end of battle and also how the variables upon which it depends vary over time. Based upon such considerations, it may be shown that there are three cases to be considered:

$$\text{Case (a)} \quad \frac{p}{q} = \frac{b_1}{b_2} ,$$

$$\text{Case (b)} \quad \frac{p}{q} > \frac{b_1}{b_2} ,$$

$$\text{Case (c)} \quad \frac{p}{q} < \frac{b_1}{b_2} .$$

We consider Case (a) first. The solution for this case is shown in Figure 1. Even though explicit expressions have not been obtained for certain model parameters, the dependence of the optimal control upon these quantities can still be qualitatively discussed. It may be shown that the optimal control depends on the state variables x_1 and x_2 (and also the attrition coefficients) in each "decision region." Above the line $a_1b_1x_1 = a_2b_2x_2$, denoted as L, the optimal control $\phi^* = 0$ is used until this line is encountered. When L is reached, the singular control $\phi^* = \frac{a_2}{a_1 + a_2}$, which keeps the trajectory on L, is used until the end of the battle at $t = T$. That

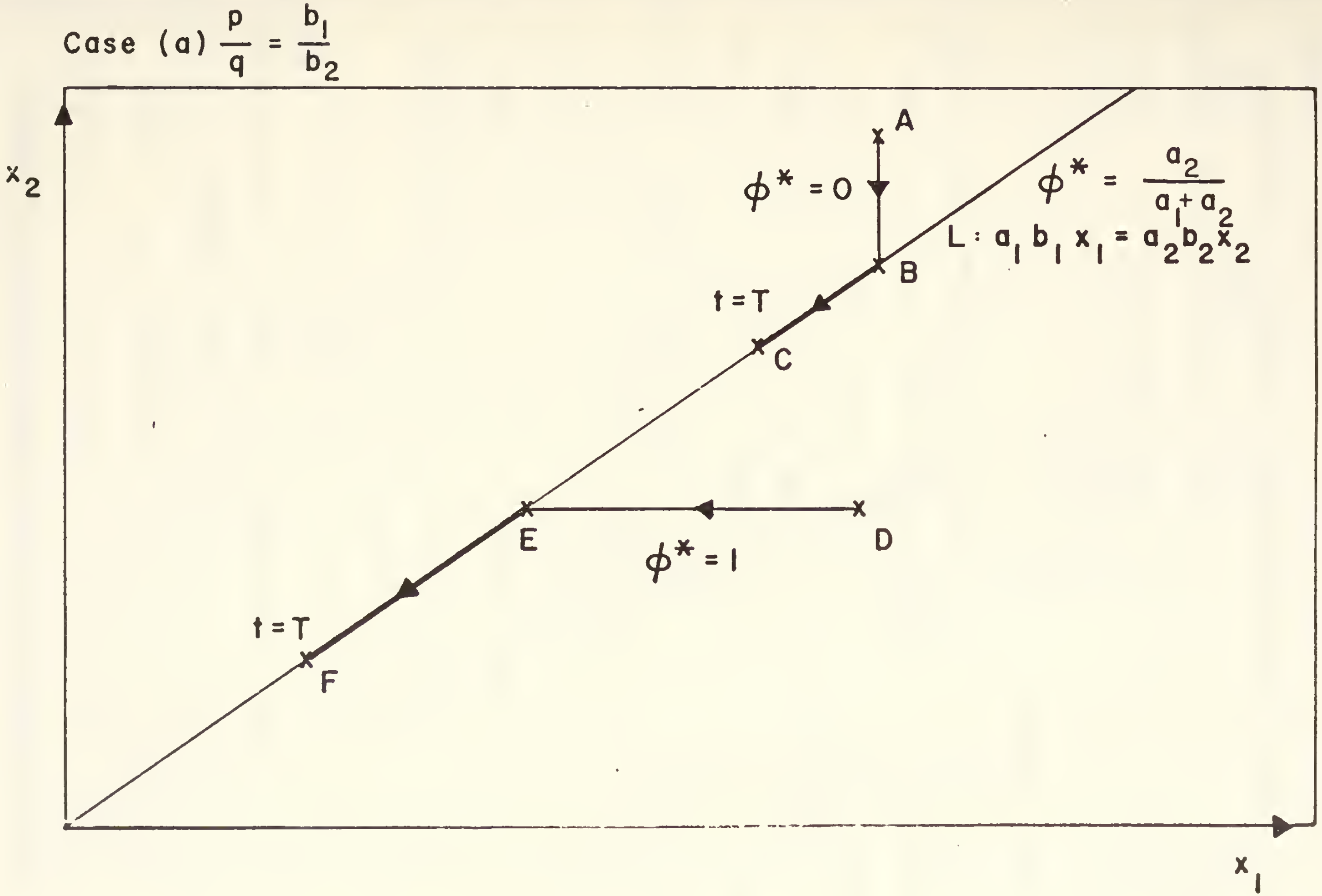

Figure 1. Optimal allocation for linear-law attrition.

portion of an optimal trajectory which lies on L (for a finite interval of time) is a singular subarc. The above type of solution holds for arbitrary initial values of x_1 and x_2: $x_1(t=0) = x_1^0$ and $x_2(t=0) = x_2^0$. The time history of the optimal control is traced for two particular initial force ratios denoted as points A and D in Figure 1. At point D, $\frac{x_1^0}{x_2^0} > \frac{a_2b_2}{a_1b_1}$ and $\phi^* = 1$ is used until the line L is encountered at point E.

The solution for Case (b) is shown diagrammatically in Figure 2. It is similar to the preceding case except that another line, L' with equation $a_1px_1 = a_2qx_2$, plays a central role besides the singular "surface" denoted as L. The final segment of that part of any optimal trajectory which lies below L' consists of the use of the optimal control $\phi^* = 1$ for the final stages of battle. Such is the case for the optimal trajectories denoted as (2), (3), and (4). Any optimal trajectory which lies entirely above L', such as (1), has a corresponding optimal control of $\phi^*(t) = 0$ for $0 \leq t \leq T$, whereas a similar remark holds for any one that lies entirely below L, such as (5). Case (c) is symmetric to Case (b).

As has been noted previously (see sections 2.1 and 2.3), the structure of the optimal allocation policies in these tactical allocation problems is dependent upon how the Y force values the surviving X-force types relative to their kill rate against the Y force. Case (a): $\frac{p}{q} = \frac{b_1}{b_2}$ above is when Y assigns utility to surviving X-force types in exact proportion to their destructive capability against Y. In this case, the optimal target selection tactic depends

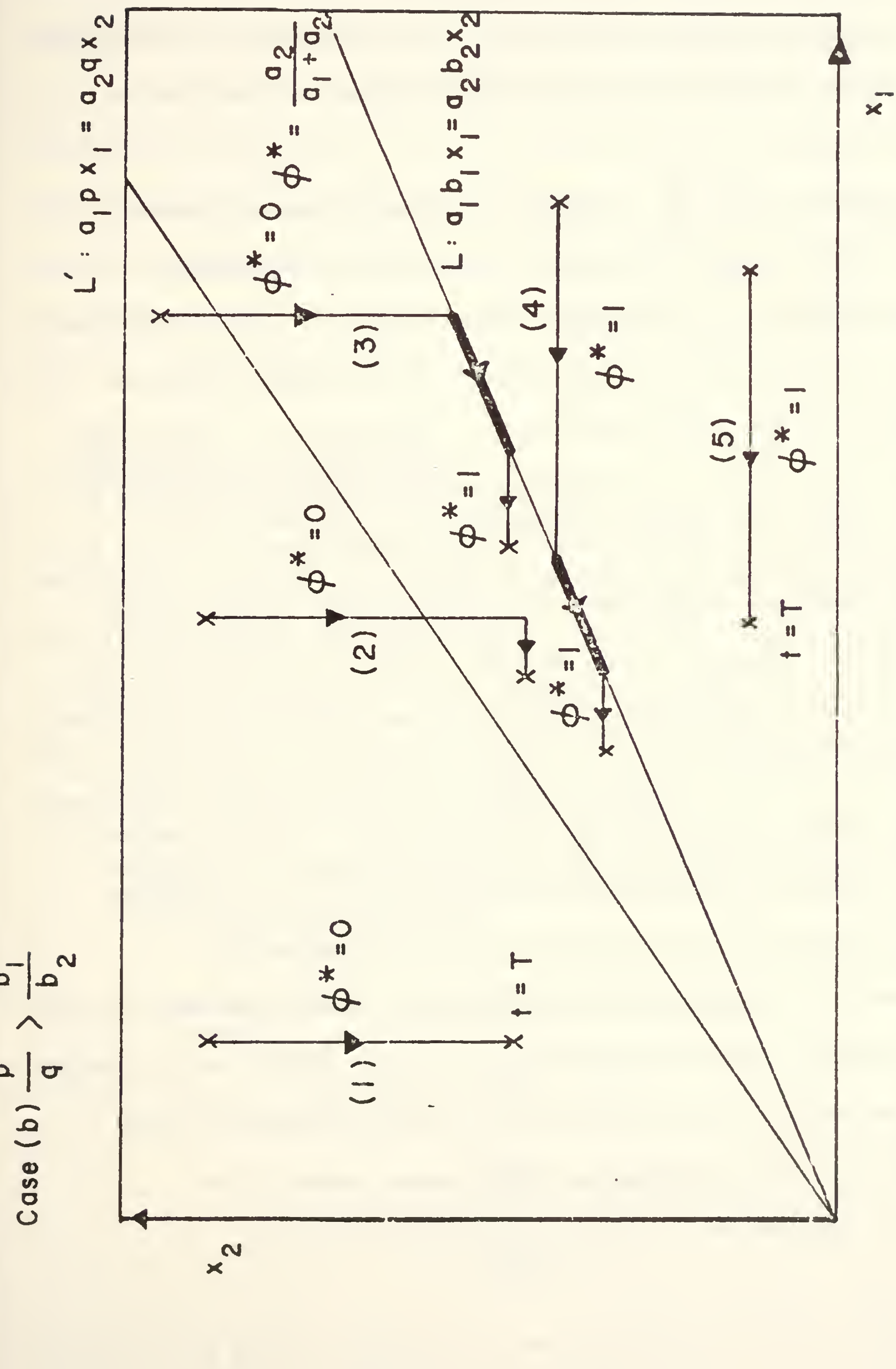

Figure 2. Optimal allocation for linear-law attrition.

only upon the state of the system as is seen with reference to Figure 1. The optimal tactic is to use $\phi^*(t) = 0$ above the line L with equation $a_1b_1x_1 = a_2b_2x_2$. The line L also represents an "equilibrium" trajectory which the system follows whenever this line is reached.

Case (b): $\frac{p}{q} > \frac{b_1}{b_2}$ is when Y assigns a greater value to surviving X_1's than in proportion to their kill rate against Y relative to that of X_2. Again, the optimal tactic depends upon the state of the system, only this dependence itself depends upon the "time phase" of the fixed length battle. There is a τ_1 such that for $0 \leq \tau \leq \tau_1$, where τ is "backwards time" measured from the end of battle $\tau = T - t$, the optimal tactic depends upon the state of the system relative to the line L' with equation $a_1px_1 = a_2qx_2$ (see Figure 2). Below L' the optimal tactic is to use $\phi^*(\tau) = 1$. For $\tau_1 \leq \tau \leq T$, the optimal tactic is to use $\phi^*(\tau) = 0$ above the line L with equation $a_1b_1x_1 = a_2b_2x_2$. We recall that the part of an optimal battle trajectory that coincides with L is a singular subarc. It may be shown that the "crossover time," τ_1, depends upon the particular battle trajectory under consideration.

Based on our examination of the above problem, we see that the structure of the optimal allocation policies for targets which undergo a linear-law attrition process has the following characteristics:

(1) possible division of fire between target types,

(2) dependence of allocation upon force levels.

These characteristics should be contrasted to those previously observed with respect to a square-law attrition process (see sections 2.1 and 2.3). When there is a linear-law attrition process for target types, we may have other than 0 or 1 as the optimal allocation policy. Also, the allocation depends upon the force levels of target types. We have attempted to explain this structure of the optimal allocation policies in terms of the nature of the attrition process in section 2.1.

3. Regulation of Firing Rate.

In this section we will examine a sequence of models of increasing complexity for which the effect of ammunition limitations on firing rate(fire discipline) will be explored. In each case, we consider two homogeneous forces engaged in combat described by a square law. Our results for these problems are of a more preliminary nature than those reported above for target selection.

3.1. Battle of Prescribed Duration with Constant Attrition Rates.

We consider the problem

$$\underset{\phi(t)}{\text{maximize}}\ px(T) - qy(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:}\quad \frac{dx}{dt} = -a_1 y$$

$$\frac{dy}{dt} = -\phi \nu a_2 x$$

$$\frac{dz}{dt} = \phi \nu$$

$$x, y \geq 0, \quad 0 \leq \phi \leq 1, \quad z(t=0) = 0 \quad \text{and} \quad z(t=T) \leq A < \nu T = \nu \int_0^T dt,$$

where ν is the maximum firing rate of each X unit. It should be noted that the nature of the attrition coefficients a_1 and a_2 is different, since a_1 has incorporated in it a firing rate.

We have assumed that each X combatant has a limited supply of ammunition (his "basic load"), denoted by A. There is no redistribution of ammunition during the course of battle, i.e., when a casualty occurs, his ammunition is lost to the other combatants. We further assume that the ammunition supply is such that a combatant could not fire at his maximum firing rate for the prescribed duration of the battle, for when $A \geq \nu T$ it is easily seen that the optimal strategy is to fire at the maximum possible rate, $\phi^*(t) = 1$ for $0 \leq t \leq T$.

The optimal regulation of firing rate turns out to be

$$\phi^*(t) = 1 \quad \text{for} \quad 0 \leq t \leq T_1 \quad \text{where} \quad T_1 = \frac{A}{\nu},$$

$$\phi^*(t) = 0 \quad \text{for} \quad T_1 \leq t \leq T.$$

This was determined as follows. The Hamiltonian is given by

$$H(t, x_i, p_i, \;) = \phi\nu(p_3 - p_2 a_2 x) - p_1 a_1 y,$$

where $p_i(t)$ is the dual variable corresponding to the i^{th} state variable (we denote x as x_1, y as x_2, z as x_3). Maximization of the Hamiltonian leads to the following control law

$$\phi^*(t) = \begin{cases} 0 & \text{for} \quad (-p_2(t)) a_2 x < \lambda^* \\ 1 & \text{for} \quad (-p_2(t)) a_2 x > \lambda^* > 0, \end{cases}$$

where $p_3(t) = -\lambda^*$. The adjoint differential equations are given by

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = \phi \nu a_2 p_2 \quad \text{with} \quad p_1(t=T) = p$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = a_1 p_1 \quad \text{with} \quad p_2(t=T) = -q$$

$$p_3(t) = \text{constant} \quad \text{with} \quad p_3(t=T) \text{ unspecified.}$$

To develop the desired solution we work backwards from the end of battle. Hence, we introduce the "backwards time" variable $\tau = T - t$ and consider a backwards integration of the state and adjoint differential equations from the fixed end of the battle, $t = T$. Thus, $\frac{dp_1}{d\tau} = -\phi\, \nu a_2 p_2$, etc. It is easy to show that $p_1(\tau)$, $x(\tau)$, and $y(\tau)$ are non-decreasing functions of τ (regardless of the value of ϕ) with $p_1(\tau=0) = p$, $x(\tau=0) = x_s$, and $y(\tau=0) = y_s$. Similarly, $p_2(\tau)$ is a strictly decreasing function of τ. Hence, $Q(\tau) = (-p_2(\tau))a_2 x$ is a strictly increasing function of τ with an initial value of $Q(\tau=0) = qa_2 x_s$. Thus, p_3 must be negative, and $\phi^*(\tau)$ never switches back to 0 once it becomes 1.

The optimal tactic for this model is disturbing, since it is not intuitively appealing to fire at one's maximum firing rate until one runs out of ammunition and to spend the final stages of battle without ammunition. However, after a little reflection we realize that it is this model's assumption of constant attrition rates which has negated the holding of one's fire as an optimal tactic. Hence, we are led to consider other models for further insight.

3.2. Battle of Prescribed Duration with Time Varying Kill Rates.

We consider the problem

$$\underset{\phi(t)}{\text{maximize}}\ px(T) - qy(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:}\quad \frac{dx}{dt} = -a_1(t)y$$

$$\frac{dy}{dt} = -\phi\nu a_2(t)x$$

$$\frac{dz}{dt} = \phi\nu$$

$x,y \geq 0$, $0 \leq \phi \leq 1$, $z(t=0) = 0$ and $z(t=T) \leq A < \nu T$.

The precise nature of $a_1(t)$ and $a_2(t)$ will depend upon the specific s under consideration. It seems reasonable to assume that in many real world situations $a_1(t)$ and $a_2(t)$ would be monotonically increasing functions of time, e.g., two forces closing with one another. All the previous solution steps remain the same except for the effect of $a_1(t)$ and $a_2(t)$ increasing with time. This may change the nature of the solution markedly, although the optimal control is still bang-bang. One can show, in fact, that a singular solution is impossible for this problem. The quantity $Q(\tau) = (-p_2(\tau))a_2(\tau)x(\tau)$ is no longer guaranteed to be a strictly increasing function of τ, since $a_2(\tau)$ is strictly decreasing (but positive). This allows the possiblity that the optimal tactic may be to hold one's fire and conserve ammunition in the early stages of battle so that $\phi^*(t=T) = 1$ at the end of battle.

The way in which ammunition is conserved depends on the specific nature of $a_1(t)$ and $a_2(t)$. As we have pointed out above, under special circumstances to solution to variable coefficient Lanchester-type equations (and also the adjoint system of equations) is no more complicated than the constant coefficient case. If we assume that $\frac{\nu a_2(\tau)}{a_1(\tau)} = \frac{k_{a_2}}{k_{a_1}}$, then there are relatively simple expressions for $p_2(\tau)$ and $x(\tau)$. For example, when $\phi = 1$ for $0 \leq \tau \leq \tau_1$, we have

$$p_2(\tau) = -q \cosh \theta(\tau) - p\sqrt{\frac{k_{a_1}}{k_{a_2}}} \sinh \theta(\tau),$$

where $\theta(\tau) = \sqrt{k_{a_1} k_{a_2}} \int_0^\tau h(\tau)d\tau$. However, even such relatively simple expressions for these quantities has not led to a general statement about under what circumstances one holds his fire even though this can be readily numerically determined for any particular set of parameter values. A question of practical importance to be answered by further investigation along these lines is, "For what types of attrition-rate range dependencies is it the optimal tactic to hold one's fire until an enemy gets closer due to an ammunition constraint."

R. Isaacs has studied some similar problems in his book <u>Differential Games</u> [15] and has explored some aspects much deeper than presented here. Isaacs tried to resolve the problem of shooting up all of one's ammunition before the end of the battle by modifying the payoff. Another approach might be to consider a terminal control problem.

3.3. Fight to the Finish with Limited Ammunition.

We consider briefly the constant coefficient problem of section 3.1 only with T unspecified and with terminal states defined by (1) $x(T) = 0$ and (2) $y(T) = 0$. In this case, the optimal tactic is dependent upon the force levels, since the ammunition constraint is not binding for

$$y_0\sqrt{\frac{a_1}{\nu a_2}} \leq x_0 \tanh\left(A\sqrt{\frac{a_1 a_2}{\nu}}\right).$$

It should be noted that the above condition also guarantees that X will win. Furthermore, the X forces in order to win are required to have enough ammunition to fire at their maximum rate during the entire duration of the battle. Hence, we see that concentration of forces reduces the ammunition requirement per man, since the length of battle is determined by the initial numbers of forces committed to battle.

3.4. A Two-Sided Extension.

It seems appropriate to contrast qualitatively the structure of the optimal regulation of firing rate for the above problems with a two-sided version, even though we do not attempt to solve the latter at this time. Thus, we consider the problem

$$\max_{\phi(t)} \min_{\psi(t)} px(T) - qy(T) \quad \text{with } T \text{ specified}$$

$$\text{subject to:}\quad \frac{dx}{dt} = -\psi a_1 \nu_1 y$$

$$\frac{dy}{dt} = -\phi a_2 \nu_2 x$$

$$\frac{du}{dt} = \phi \nu_2$$

$$\frac{dv}{dt} = \psi \nu_1$$

$$x,y \geq 0,\quad 0 \leq \phi,\quad \psi \leq 1,\quad u(t=0) = 0,\quad u(t=T) \leq A_2 < \nu_2 T,$$
$$v(t=0) = 0,\quad v(t=T) \leq A_1 < \nu_1 T.$$

Unlike the previous one-sided version of this problem, it is now possible to have $\phi^*(t=T) = 1$ with limited ammunition. This possibility has arisen since the Y forces may hold their fire during the early stages of engagement. Questions now arise as to the advantage of delivering the first shot, e.g., is there a time lag before fire is returned?, and we move into the realm of games of timing studied at RAND [19].

4. Extensions to Differential Games.

Even though it is certainly true that combat is an environment of conflicting interests in which the potential actions of both friendly and enemy forces must be considered, there is much to be learned from one-sided dynamic optimization models. We view these simplified idealizations presented here as "building blocks" for more sophisticated scenarios. We feel, however, that an understanding of the structure of optimal tactics for these initial models

is essential before one continues his examination of a sequence of models of greater and greater complexity. Hence, it seems appropriate to review the intimate connection between optimal control theory and differential games.

It has been stated that optimal control problems may be viewed as one-sided differential games for which the roles of all but one of the competing players have been suppressed [3]. A concise discussion of the inter-relationships between these two subjects is contained in Y. C. Ho's [14] excellent review of Isaacs book [15] (see also Chapter 9 in [9]).

When one recalls the equivalence of variational problems and partial differential equations of the first order (first pointed out by C. Jacobi nearly 140 years ago [17]), the relationship between optimal control and differential games may be viewed as follows. In an optimal control problems we are seeking the solution to the Hamilton-Jacobi-Bellman equation for the optimal value function, denoted as S (sometimes referred to as Hamilton's characteristic function in the calculus of variations literature [23])

$$\frac{\partial S}{\partial t} + \underset{\phi(t)}{\text{maximum}}\ H(t, x_i, \frac{\partial S}{\partial x_i}, \phi) = 0$$

with appropriate boundary conditions. In a differential game we seek the solution to the Bellman-Isaacs equation [13]

$$\frac{\partial S}{\partial t} + \underset{\phi(t)}{\text{maximum}}\ \underset{\psi(t)}{\text{minimum}}\ H(t, x_i, \frac{\partial S}{\partial x_i}; \phi, \psi) = 0.$$

Moreover, this approach may also be extended to when stochastic effects are present as has been done by S. Dreyfus [12] and others [11], [33].

It also seems appropriate to mention the relationship of dynamic programming to these techniques. Consideration of the equation satisfied by the optimal value function points out clearly an important aspect of dynamic programming, its being a discrete approximation technique for solving variational problems [12]. It is, however, a dual approach which generates an optimal trajectory as an envelope of tangents rather than as a sequence of points [1]. The value of these continuous models lies in their mathematical tractability which many times allows us to develop "closed-form" solutions. This should be contrasted with dynamic programming models for which general solutions are rarely obtained, and one must be satisfied with generating a numerical solution for a particular set of parameter values. In this case, it is difficult (if not impossible) to see the structure of optimal allocation policies and its dependence upon model form without a parametric analysis of model output.

We have previously pointed out [26] that the existing theory of differential games is only applicable to problems with pure strategy solutions. The structure of two-sided fire distribution problems in the Lanchester theory of combat as formulated by H. Weiss [31] leads to such a result. However, when defensive capabilities were considered in the attrition process in a tactical air war game extensively studied at RAND, the resulting model did not possess a solution in pure

strategies [2], [4], [5]. Thus, there are limits to the applicability of such variational models.

We have, therefore, used these optimal control problems to study many aspects of corresponding two-sided variational problems: the effect of different boundary conditions, devising solution procedures, study of singular behavior, differences in the structure of optimal allocation policies for various model forms. Most solution aspects of the one-sided problem are present in the two-sided one. There are some significant differences, however. In solving the supporting weapon system game of H. K. Weiss [32], we have encountered solution behavior unique to terminal control attrition games: there may exist a domain of controllability for a given terminal state but entry to this state may be "blockable" by the "losing" player [28]. In other words, there is a path determined by the necessary conditions leading from each point in a region of the initial state space to a terminal state, but the "losing" player may use a strategy other than his extremal strategy for this path to actually win. We made use of our knowledge of a related optimal control problem [26] to solve this differential game.

5. <u>Implications of Models</u>.

It seems appropriate to briefly discuss the general implications in the following areas of the models examined in this appendix:

(1) optimal tactical allocation,

(2) intelligence,

(3) command and control systems,

(4) human decision making.

Our discussion of these areas is not mutually exclusive.

Of interest to the military tactician is whether target selection rules evolve dynamically during the course of battle. Are target priorities static or do they evolve dynamically with the course of battle? With respect to optimal control models, this may be mathematically stated as whether there are transition (switching) surfaces in the solution. We have seen in the idealized and simplified models studied here that target priorities do change. This is related to the evolution of marginal return of target destruction (value of dual variable). We have seen that this evolution depends on the goals of the combatants (utility assigned to surviving force types at the end of the battle) and also the conditions which terminate the battle. In the terminal control problem studied here, a shift in target priorities is present only in a losing case, whereas in a fixed duration battle such a switch is independent of winning or losing but depends only on weapon system capabilities and the prescribed duration of battle.

Even though these models assume complete and instantaneous information, it appears that some inferences may be made for cases where uncertainty is present. In the terminal control case, we saw that selection of tactics depends on a knowledge of the enemy's strength and capabilities, since the terminal state of combat must be determined before optimal strategies can be. For a battle of prescribed duration, e.g., fighting a delaying action in a regrograde movement to protect the withdrawal of troops, tactics depend only on enemy and friendly

capabilities and length of combat, not the initial force levels. For such cases the estimate of combat length is critical, since changes in target priorities are determined relative to the end of the engagement.

Schreiber [24] has proposed an idealized and simple, but yet illuminating, way of quantitatively showing the value of intelligence and command control capabilities. He introduces the concept of "command efficiency," which is measured by the fraction of the enemy's destroyed units from which fire has been redirected. The effect of poor intelligence and poor capabilities for redirecting fire from destroyed targets is to produce "overkill." Schreiber's equations for combat involved this fraction called "command efficiency," and they reduce to Lanchester-type equations for area fire when the fraction is 0 and aimed fire for a value of 1. We have seen that the optimal tactics are quite different for these two cases. When intelligence and command control systems are very efficient, the optimal tactic is seen always to be concentration of fire on a specific target type. When capability for redirection of fire from destroyed targets is poor (either through damage assessment or constraints on new target acquisition), the optimal tactic may be to allocate fire in a proportional fashion over target types in a way that holds the ratios of target density in each target area to be constant. Another implication is that supporting weapon systems (e.g., artillery) concentrate fire on selected point targets, but that it sometimes is best to allocate fire proportionately over various area targets these models suggest that the tactics of target engagement may vary with command and control capabilities.

These models also show the importance of intelligence in devising the "best" tactics in combat. Intelligence on enemy weapon system capabilities (kill rates including target acquisition rates) and potential length of engagement play a central part. We also have seen that for fights to the finish and linear law attrition cases intelligence on enemy force levels is also required. For artillery fire support missions against various troop concentrations, knowledge of troop densities is essential in the assignment of target priorities. Particularly dense concentrations where the initial kill potential is high are seen to be cases where the optimal tactic is to concentrate fire on one target for awhile.

Another argument for the concentration of forces is seen to emerge from the study of these simplified models. When ammunition is limited, a concentration of forces has the effect of counter-balancing this constraint. For example, in a fire fight numerical superiority could mean that the enemy force level would be reduced such that he would disengage in time before the friendly ammunition restriction became critical.

These models may be interpreted to show the value of human judgment in combat. They indicate, as does common sense and experience, that in battle a commander must use his judgment to ascertain to what end can the course of battle be steered so that he may devise his strategy accordingly. The demonstrated sensitivity of these models to many factors shows the importance of human assessment of a situation and the importance of good judgment in assigning utility to forces surviving the battle at hand.

6. Summary.

The results of this appendix may be summarized as follows:

(1) a sequence of one-sided models has been presented which shows that the tactics of target selection may be sensitive to force strengths, target acquisition process, the type of attrition process, and/or the termination conditions of combat,

(2) a sequence of models have been presented which shows some preliminary results about the effect of resource constraints on firing discipline and concentration of forces,

(3) tactics for target selection are heavily dependent upon "command efficiency,"

(4) concentration of fire always on one target type among many occurs as an optimal tactic only when target acquisition is not subject to diminishing returns,

(5) target priorities don't change over time when one assigns a worth to surviving target types in direct proportion to their kill rate against you.

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Appendix C. Target Selection in Lanchester Combat: Heterogeneous Forces and Time Dependent Attrition Rates.

1. Introduction.

In this appendix we develop solutions to a sequence of models for optimal target selection in combat against heterogeneous enemy forces. The combat continues over a period of time with a choice of tactics available to one side and subject to change with time. Optimal target selection rules are developed through the combination of Lanchester-type equations for combat attrition and deterministic optimal control theory (Pontryagin maximum principle). Both constant attrition-rate coefficients and some special cases of variable attrition rates are considered. The synthesis of optimal target engagement policies was facilitated by exploiting special mathematical structures in these problems.

Many years ago, Karl von Clausewitz said that if theory caused a more critical study of war, then it had achieved its purpose. In Appendix B we studied the structure of the optimal allocation policies for some basic elementary tactical situations described by Lanchester-type equations of warfare. We did this by, among other things, contrasting the "best" target selection policies for a sequence of scenarios: prescribed-duration battle, terminal-control battle, two enemy target types, many enemy target types, etc.

In Appendix B we frequently summarized results and promised the details of solution development (for the non-trival cases) to be available elsewhere.

Thus, the purpose of this appendix is to provide supporting analysis details for target selection in Lanchester combat against heterogeneous enemy force types and/or combat in which the attrition rates may change over time. All battle scenarios that we consider here are for a prescribed duration of time. We also present some more general results than those reported in Appendix B: target selection for combat against several enemy force types with all attrition rates subject to change over time.

Again, the purpose of this appendix is to provide analysis details. The interested reader will find an extensive discussion of the structure of optimal allocation policies and model implications in Appendix B. Our presentation in this present appendix follows the historical order in which we attacked these problems. We could have merely reported the solution of the problem in section 4. This would have been more concise. We feel, however, that it is more valuable to show the reader how we have used insight gained from simpler problems to provide guidance for solving more complex ones and generalizing results.

In this appendix we develop the solutions to all problems by the mathematical theory of optimal control. For the problems at hand we need only make use of the Pontryagin maximum principle [8].

The reader should note that there is a sign difference between developments in this country (see p. 108 of [6] or pp. 12-14 of [7]) and those in the Soviet Union [1], [8], although both approaches yield exactly the same results in applications.

We wish to impart to the reader through our examination of these problems our experience that some ingenuity is required to solve applied problems. By taking advantage of the structure to some problems, we have found that one can frequently synthesize the optimal control to a problem which at first sight appears to be too complex to allow this. By the synthesis of the optimal control, we mean the explicit determination of the time history of the optimal control from initial to terminal time. For the problems at hand, this is accomplished by determining a control law by the maximum principle and then working backwards from the end of the problem by a backwards integration of the adjoint system of differential equations for the dual variables which have a boundary condition there.

The organization of this appendix is as follows. First, we consider the problem of optimal selection of a target type from among two alternatives for some special cases of variable attrition rates. Then, we treat the more general case of several target types for some special instances of variable attrition rates. Finally, we make some observations on the results presented in this appendix.

2. Two Target Types, Some Special Cases of Variable Attrition Rates.

We consider the following prescribed duration battle:

$$\max_{\phi(t)} \quad ry(T)-px_1(T) - qx_2(T) \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1(t)y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2(t)y,$$

$$\frac{dy}{dt} = -b_1(t)x_1 - b_2(t)x_2,$$

$$x_1, x_2 y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1, \tag{1}$$

where

p, q and r are values placed on surviving force types,

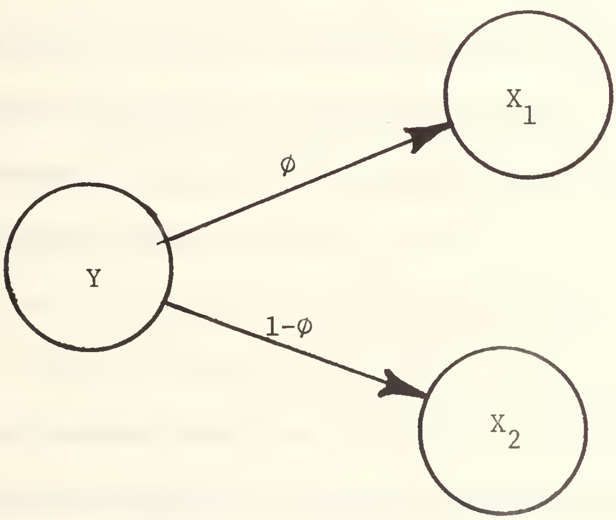
x_1, x_2 and y are average force strengths,

$a_1(t)$, $a_2(t)$, $b_1(t)$ and $b_2(t)$ are attrition rates which are allowed to change over time,

and ϕ is the fraction of Y-fire directed at X_1.

The physical interpretation of this idealized military situation is as follows: combat for a known length of time between an X force composed of two types of weapon systems (for example, infantry riflemen and machine gunners) and a homogeneous Y force (for example, riflemen only). The objective of the Y-force commander is to maximize the military worth of his surviving forces at the end of a battle lasting a known length of time T and to

minimize that of the enemy's. This is accomplished by his choice of the fraction of fire, ϕ, directed at the X_1 forces. A schematic of this scenario is shown below. It is assumed that a military worth can be assigned to survivors of each force-type and that this utility is additive. We denote this utility per unit of weapon system as p, q and r for the X_1, X_2 and Y forces, respectively.



Our state equations, i.e., the differential equation constraints, represent the attrition process in this idealized battle. We shall refer to attrition as being a "square-law" process when the casualty rate is proportional to only the number of enemy firers and as being a "linear-law" process when it is proportional to the product of the number of enemy firers and remaining targets.

For the problems in this paper we assume a "square-law" process. The interested reader can find discussions of the physical assumptions which lead to this type of attrition in articles by H. Weiss [12] and H. Brackney [5]. The essential points are that new targets are acquired at a rate independent of force levels and that fire is aimed at point targets. More recently S. Bonder [3], [4] has developed formulas for estimating the Lanchester attrition-rate coefficient, the rate at which one unit of weapon system destroys enemy targets, as a synthesis of the following factors: hit probabilities, rates of fire, target acquisition rate, weapon system projectile-target lethality characteristics, adjustment process.

In general, the Lanchester attrition-rate coefficient is range dependent, and the above formulation (1) takes this into account by having the attrition coefficients depend upon time. S. Bonder [2] has done the pioneering work on analyzing dynamic combat situations with Lanchester-type equations and variable attrition rates (also see [9], [10] for more background). For example, consider a mobile attack on a static defensive position. The effectiveness (Lanchester attrition-rate coefficient) of weapons systems depends upon force separation, and this is, in turn, related to time via the attack velocity. In [9] we showed how to develop Lanchester-type equations with either time or range as the independent variable and noted the equivalence of the two formulations (see [10] for more extensive results with variable attrition rates). Hence, the

above was our motivation for studying scenarios with time-dependent attrition rates.

We finally note that when fire is uniformly distributed over a target area [12] or the rate of target destruction is constrained by the rate of acquisition of new targets and this is inversely proportional to target density [5], a "linear-law" attrition process results. We have elsewhere [11] pointed out that this type of attrition structure leads to a fundamentally different structure for the optimal target-type selection policies.

We now develop the solution to the above problem (1). In all cases we consider the special instance when both X-force weapon systems are such that

$$b_1(t) = k_{b_1} h(t) \quad \text{and} \quad b_2(t) = k_{b_2} h(t). \tag{2}$$

This physically means that both weapon systems have the same type of range capability (for example, quadratic dependence of kill rate on range) and the same effective range, although one weapon system dominates the other in exactly the same manner at all ranges.

It seems appropriate to point out that there is a special instance when the optimal allocation policy takes a particularly simple form. This is when the Y force values surviving X-force types in direct proportion to their kill rate against the Y force, i.e., $p/q = k_{b_1}/k_{b_2} = b_1(t)/b_2(t) = b_1(t=T)/b_2(t=T)$. The optimal allocation rule for $0 \leq t \leq T$ is then

$$\phi^*(t) = \begin{cases} 1 & \text{for } a_1(t)b_1(t) > a_2(t)b_2(t), \\ 0 & \text{for } a_1(t)b_1(t) < a_2(t)b_2(t). \end{cases} \tag{3}$$

This was obtained by application of optimal control theory [6] as follows.

The Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = -p_1(t)\phi a_1(t)y - p_2(t)(1-\phi)a_2(t)y$$
$$- p_3(t)\{b_1(t)x_1 + b_2(t)x_2\}, \tag{4}$$

where the dual variables, $p_i(t)$ for $i = 1,2,3$, are the effect on the total value of survivors at the end of battle when an optimal policy is followed from time t until time T of having an additional unit of $x_i(t)$ [here we have let $y(t) = x_3(t)$]. Another way to state this is that

$$p_i(t) = \frac{\partial S}{\partial x_i}(t) \quad \text{where} \quad S = S(t,x_1,x_2,y) \quad \text{is}$$

the optimal value function. According to the maximum principle, the optimal control is determined by

$$\underset{0 \le \phi \le 1}{\text{maximize}}\ H(t,x_i,p_i,\phi), \tag{5}$$

and this leads to

$$\phi^*(t) = \begin{cases} 1 & \text{for } v(t) > 0, \\ 0 & \text{for } v(t) < 0, \end{cases} \tag{6}$$

where

$$v(t) = -p_1(t)a_1(t) + p_2(t)a_2(t). \qquad (7)$$

The adjoint system of differential equations for the dual variables is given by

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x_1} = b_1(t)p_3 \qquad \text{with} \quad p_1(t=T) = -p,$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial x_2} = b_2(t)p_3 \qquad \text{with} \quad p_2(t=T) = q,$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial x_3} = \phi a_1(t)p_1 + (1-\phi)a_2(t)p_2 \quad \text{with} \quad p_3(t=T) = r. \qquad (8)$$

By assumption (2) and the adjoint system (8), it is readily shown that

$$\frac{dp_1}{dp_2} = \frac{k_{b_1}}{k_{b_2}},$$

which, when integrated, leads to

$$p_1(t) = \frac{k_{b_1}}{k_{b_2}} \{p_2(t) + q\} - p. \qquad (9)$$

Using (9), we may write $v(t)$ defined by (7) as follows

$$v(t) = \frac{1}{k_{b_2}}\left[(-p_2(t))\{k_{b_1}a_1(t) - k_{b_2}a_2(t)\} + k_{b_2}q\ \{\frac{p}{q} - \frac{k_{b_1}}{k_{b_2}}\}\ a_1(t)\right]. \qquad (10)$$

Assuming that $p/q = k_{b_1}/k_{b_2}$, we see that the optimal target selection rule as given by (3) is an immediate consequence of (6), (7) and (10). We have also made use of the fact that $p_2(t) < 0$, which is easily shown.

We now consider the case when

$$a_1(r) = k_{a_1} h(t) \quad \text{and} \quad a_2(t) = k_{a_2} h(t). \tag{11}$$

This means that all four attrition rates are proportional to the same basic time dependence. As we have pointed out elsewhere, both the solution to Lanchester-type equations [9] and the solution to the fire distribution problem [11] in this instance are essentially the same as for constant coefficients, only with a transformation of the time scale. The solution is shown in Table I.

We develop the solution shown in Table I for the nonrestrictive assumption that $k_{a_1} k_{b_1} > k_{a_2} k_{b_2}$ as follows. Consider the quantity $v(t)/h(t)$ which by (7) is seen to be given by

$$\frac{v(t)}{h(t)} = -p_1(t)k_{a_1} + p_2(t)k_{a_2}. \tag{12}$$

Differentiation of (12) leads to

$$\frac{d}{dt}\left(\frac{v(t)}{h(t)}\right) = -h(t)(k_{a_1}k_{b_1} - k_{a_2}k_{b_2})p_3, \tag{13}$$

where we have made use of both the adjoint equations (8) and also assumption (2). Observing that $h(t) > 0$, it is clear that the control law (6) is equivalent to

TABLE I. Solution to Target Selection Problem

Battle of Prescribed Duration with Variable Attrition Rates

Special assumption: $a_1(t)/a_2(t) = k_{a_1}/k_{a_2}$, $b_1(t)/b_2(t) = k_{b_1}/k_{b_2}$, and $a_1(t)/b_1(t) = k_{a_1}/k_{b_1}$

Nonrestrictive assumption: $k_{a_1}k_{b_1} > k_{a_2}k_{b_2}$

Case A: $k_{a_1}p \geq k_{a_2}q$	$\phi^*(t) = 1$ for $0 \leq t \leq T$
Case B: $k_{a_1}p < k_{a_2}q$	(a) for $\tau_1 > T$ $\phi^*(t) = 0$ for $0 \leq t \leq T$ (b) for $\tau_1 < T$ $\phi^*(t) = 1$ for $0 \leq t \leq T - \tau_1$ $\phi^*(t) = 0$ for $T - \tau_1 \leq t \leq T$

Note: τ_1 is determined from the transcendental equation.

$$r\sqrt{\frac{k_{b_2}}{k_{a_2}}}\sinh\{\sqrt{k_{a_2}k_{b_2}}\int_0^{\tau_1} h(\tau)d\tau\} + q\cosh\{\sqrt{k_{a_2}k_{b_2}}\int_0^{\tau_1} h(\tau)d\tau\} = \frac{k_{a_1}(k_{b_1}q - k_{b_2}p)}{(k_{a_1}k_{b_1} - k_{a_2}k_{b_2})}$$

$$\phi^*(t) = \begin{cases} 1 & \text{for } \dfrac{v(t)}{h(t)} > 0, \\ 0 & \text{for } \dfrac{v(t)}{h(t)} < 0, \end{cases} \tag{14}$$

Since we develop the solution to this problem by working backwards from the end $t = T$, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T - t$. Observing that $\frac{d}{dt} = -\frac{d}{d\tau}$, we have that

$$\frac{d}{d\tau}\left(\frac{v(\tau)}{h(\tau)}\right) = h(\tau)(k_{a_1}k_{b_1} - k_{a_2}k_{b_2})p_3(\tau). \tag{15}$$

It is easily shown that $p_3(t) > 0$ for all t. Hence, it follows from (15) that

$$\frac{d}{d\tau}\left(\frac{v(\tau)}{h(\tau)}\right) > 0 \quad \text{for all} \quad \tau, \tag{16}$$

since $h(t) > 0$ and we have assumed that $k_{a_1}k_{b_1} > k_{a_2}k_{b_2}$.

To solve this problem, we work backwards from the end of battle at $t = T$, or equivalently $\tau = 0$. At this point we have

$$v(t{=}T) = h(t{=}T)(pk_{a_1} - qk_{a_2}). \tag{17}$$

Thus, we have two cases to consider.

In Case (A): $k_{a_1}p \geq k_{a_2}q$, we have that $v(t{=}T) \geq 0$ by (17). Consideration of (16) immediately yields that $\frac{v(\tau)}{h(\tau)} > 0$ for all $\tau > 0$ and thus $\phi^*(t) = 1$ for $0 \leq t \leq T$ by (14).

In Case (B): $k_{a_1}p < k_{a_2}q$, we have that $v(t{=}T) < 0$ by (17) and the assumption that $h(t{=}T) > 0$ (the battle ends with non-zero attrition rates). Thus

$$\frac{v(\tau=0)}{h(\tau=0)} < 0,$$

with

$$\frac{d}{d\tau}\left(\frac{v(\tau)}{h(\tau)}\right) > 0 \quad \text{for all} \quad \tau.$$

Thus, $\frac{v(\tau)}{h(\tau)}$ must change from a negative quantity to a positive one at some time. Let τ_1 denote the time when this happens. Then for $0 \leq \tau < \tau_1$ we have that $\phi^*(\tau) = 0$. We can easily determine $p_2(\tau)$ during this interval by considering the adjoint equations (8) which may be written as

$$\frac{dp_2}{d\tau} = -b_2(\tau)p_3(\tau) \qquad \text{with} \quad p_2(\tau=0) = -q,$$

and

$$\frac{dp_3}{d\tau} = -a_2(\tau)p_2(\tau) \qquad \text{with} \quad p_3(\tau=0) = r. \tag{18}$$

These variable coefficient adjoint equations are readily integrated, however, by consideration of (2) and (11), i.e., $a_2(\tau)/b_2(\tau) =$ constant, and the observations elaborated upon in [10], i.e., from (18) we can obtain the following linear second order differential equation for $p_2(\tau)$

$$\frac{d^2p_2}{du^2} - \frac{a_2(\tau)}{b_2(\tau)}\, p_2 = 0,$$

with

$$p_2(u=0) = -q,$$

$$\frac{dp_2}{du}(u=0) = -r,$$

and

$$u = \int_0^{\tau} b_2(\tau)d\tau.$$

Hence, we readily find that

$$p_2(\tau) = -q \cosh \theta(\tau) - r \sqrt{\frac{k_{b_2}}{k_{a_2}}} \sinh \theta(\tau), \tag{19}$$

where

$$\theta(\tau) = \sqrt{k_{a_2} k_{b_2}} \int_0^{\tau} h(\tau)d\tau. \tag{20}$$

Substituting (19) into (10), we obtain that

$$v(\tau) = \frac{h(\tau)}{k_{b_2}} \Bigg[\{q \cosh \theta(\tau) + r \sqrt{\frac{k_{b_2}}{k_{a_2}}} \sinh \theta(\tau)\}(k_{a_1} k_{b_1} - k_{a_2} k_{b_2}) + k_{a_1} k_{b_2} q\{\frac{p}{q} - \frac{k_{b_1}}{k_{b_2}}\} \Bigg], \tag{21}$$

where $\theta(\tau)$ is given above by (20). Observing that τ_1 is determined by the transcendental equation $v(\tau_1) = 0$, we see that we have obtained all the results presented in Table I.

3. <u>Several Target Types, Constant Attrition Rates.</u>

We consider the following prescribed duration battle:

$$\underset{\phi_i(t)}{\text{maximize}} \quad vy(T) - \sum_{i=1}^{n} w_i x_i(T) \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = -\phi_i a_i y \quad \text{for } i = 1,\ldots,n,$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i x_i,$$

$$x_i, y \geq 0, \quad \phi_i \geq 0 \quad \text{for } i = 1,\ldots,n \quad \text{and} \quad \sum_{i=1}^{n} \phi_i = 1,$$

where v and w_i for $i = 1,\ldots,n$ are values placed on surviving specific force types and all other symbols are used in the same sense as above. We now present the details behind the solution which we stated in a companion paper [11].

The Hamiltonian to the above problem is given by

$$H(t,x_i,p_i,\phi_i) = -y \sum_{i=1}^{n} a_i p_i(t)\phi_i - p_{n+1} \sum_{i=1}^{n} b_i x_i. \tag{22}$$

According to the maximum principle, the optimal control (there is only one extremal) is determined by the (trivial)linear program

$$\underset{\phi_i}{\text{maximize}} \quad H(t,x_i,p_i,\phi_i)$$

$$\text{subject to:} \quad \sum_{i=1}^{n} \phi_i = 1,$$

$$\phi_i \geq 0,$$

which in turn leads to

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} a_i(-p_i(t))\phi_i \\ &\text{subject to:} \quad \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad\qquad\qquad \phi_i \geq 0. \end{aligned} \tag{23}$$

By inspection the solution to (23) is easily seen to be

$$\phi_i^*(t) = \delta_{ij(t)}, \tag{24}$$

where δ_{ij} is the Kronecker delta and is equal to 1 for $i = j$ and zero otherwise and $j(t)$ is the index such that

$$a_j p_j(t) = \text{minimum } (a_1p_1, a_2p_2, \ldots, a_np_n).$$

To trace the history of ϕ_i^* over time, we must consider the adjoint system of differential equations given by

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = p_{n+1}b_i \quad \text{with} \quad p_i(t=T) = -w_i \quad \text{for} \quad i = 1,\ldots,n,$$

and

$$\frac{dp_{n+1}}{dt} = -\frac{\partial H}{\partial y} = \sum_{i=1}^{n} \phi_i a_i p_i \quad \text{with} \quad p_{n+1}(t=T) = v. \tag{25}$$

Considering (4), it is easily seen that

$$\frac{dp_i}{dp_n} = \frac{b_i}{b_n},$$

so that

$$p_i(t) = \frac{b_i}{b_n}\{p_n(t) + w_n\} - w_i. \tag{26}$$

Substituting (26) into (23), we obtain after some manipulation that the optimal control is determined by

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} c_i(t)\phi_i \\ &\text{subject to:} \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad\qquad \phi_i \geq 0, \end{aligned} \tag{27}$$

where

$$c_i(t) = \frac{(-p_n(t))}{b_n} a_i b_i \left[1 + \frac{w_i}{(-p_n(t))} \left\{ \frac{b_n}{b_i} - \frac{w_n}{w_i} \right\} \right]. \tag{28}$$

Hence, we see that if $w_i = kb_i$ for $i = 1,\ldots,n$, i.e., Y values enemy survivors in direct proportion to their kill rate against his forces, the above problem is equivalent to

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} \phi_i a_i b_i \\ &\text{subject to:} \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad\qquad \phi_i \geq 0, \end{aligned}$$

where we have made use of the easily verified fact that $p_n(t) < 0$ for all time. Thus the optimal control is given by

$$\phi_i^*(t) = \delta_{ij} \quad \text{for} \quad 0 \leq t \leq T, \tag{29}$$

where j is the index such that

$$a_j b_j = \text{maximum } (a_1 b_1, \ldots, a_n b_n).$$

Hence, for this problem when one values enemy survivors in direct proportion to their kill rate against you, the optimal tactic is to concentrate all fire on a single target type until it is entirely destroyed.

We now consider the more complex case where one does not value enemy survivors in direct proportion to their kill rate against you. We shall see that if the battle lasts long enough there will be one or more switches in the ranking of target priorities. As before, since we develop the solution to this problem by working backwards from the end $t = T$, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T = t$. At the end of battle $t = T$, we arrange the enemy target types so that n is the index such that

$$a_n w_n = a_n(-p_n(\tau=0)) = \text{maximum } (a_1 w_1, \ldots, a_n w_n). \tag{30}$$

By (23) it is easily seen that

$$\phi_i^*(t=T) = \delta_{in}. \tag{31}$$

By straightforward continuity arguments, it is readily seen that

$$\phi_i^*(\tau) = \delta_{in} \quad \text{for} \quad \tau \in [0, \tau_1), \tag{32}$$

where τ_1 is the "backwards time" of the first switch in target selection. Giving consideration to (32) and observing that

$\frac{d}{dt} = -\frac{d}{d\tau}$, we see that for $\tau \in [0,\tau_1]$ we need only consider the following equations from the adjoint system (25)

$$\frac{dp_n}{d\tau} = -b_n p_{n+1} \quad \text{with} \quad p_n(\tau=0) = -w_n,$$

$$\frac{dp_{n+1}}{d\tau} = -a_n p_n \quad \text{with} \quad p_{n+1}(\tau=0) = v, \tag{33}$$

and we recall that (re-writing (26))

$$p_i(\tau) = \frac{b_i}{b_n}\{p_n(\tau) + w_n\} - w_i \quad \text{for} \quad i = 1,\dots,n. \tag{34}$$

The above initial value problem (33) is routinely solved to yield

$$p_n(\tau) = -w_n \cosh \sqrt{a_n b_n}\,\tau - v\sqrt{\frac{b_n}{a_n}} \sinh \sqrt{a_n b_n}\,\tau. \tag{35}$$

We will now determine what conditions are necessary for a change in target selection and the time at which the change occurs, τ_1. To do this we re-write (27) and (28) as

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} e_i(\tau)\phi_i \\ &\text{subject to:} \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad\qquad \phi_i \geq 0, \end{aligned} \tag{36}$$

where

$$e_i(\tau) = a_i w_i[1 + \frac{b_i}{w_i b_n}\{(-p_n(\tau)) - w_n\}]. \tag{37}$$

We switch at the smallest τ for which

$$a_i w_i \left[1 + \frac{b_i}{w_i b_n} \{(-p_n(\tau)) - w_n\} \right] = a_n(-p_n(\tau)), \tag{38}$$

where $i = 1,\ldots,n-1$ and certain other conditions (to be determined presently) are met. Let k be the index of the target type to which fire is first shifted in "backwards time." Observe that at $\tau = 0$, we have

$$a_i w_i < a_n w_n, \tag{39}$$

for $i = 1,\ldots,n-1$, since the index n has been defined by (30). Then for $\tau_1 < \tau < \tau_2$ where τ_2 is the "backwards time" of the second switch in target selection, we have that $\varphi_i^*(\tau) = \delta_{ik}$, and thus we must have by (36) and (37)

$$a_k w_k \left[1 + \frac{b_k}{w_k b_n} \{(-p_m(\tau)) - w_n\} \right] > a_n(-p_n(\tau)),$$

which may be rearranged to yield

$$a_k(b_k w_n - b_n w_k) < (a_k b_k - a_n b_n)(-p_n(\tau)). \tag{40}$$

We now show that a necessary condition for fire to be shifted from target type n to target type k when one works backwards from the end is that $a_k b_k > a_n b_n$. The proof is as follows. We shall show that $a_k b_k \leq a_n b_n$ leads to a contradiction. First, we consider the special case when $a_k b_k = a_n b_n$. In this case (40) reduces to

$$\frac{w_n}{w_k} < \frac{b_n}{b_k} = \frac{a_k}{a_n},$$

or

$$a_n w_n < a_k w_k .$$

But this is a contradiction to (39) which must hold with $i = k$. In the case when $a_n b_n > a_k b_k$, then using the fact that $(-p_n(\tau)) > w_n$ for $\tau > 0$, we may write (40) as

$$a_k \frac{(b_n w_k - b_k w_n)}{a_n b_n - a_k b_k} > (-p_n(\tau)) > w_n,$$

but this leads to $a_k w_k > a_n w_n$ which is a contradiction to (39) as before.

Thus, $a_k b_k > a_n b_n$ and the switch in target selection occurs at

$$a_k \frac{(b_k w_n - b_n w_k)}{a_k b_k - a_n b_n} = (-p_n(\tau = \tau_1)) > 0, \tag{41}$$

so that a second necessary condition is

$$\frac{b_k}{b_n} > \frac{w_k}{w_n} .$$

In other words, we switch to fire at earlier times in the battle on the target type which causes attrition proportionally more than the ratio of values placed on survivors from the target which yields the greatest direct return at the end of battle.

To recapitulate the above, the target to which fire is first shifted (working backwards from the end of battle) has index k determined by

$$R_k = \underset{\substack{R_i>0 \\ a_ib_i>a_nb_n}}{\text{minimum}} (R_1,\ldots,R_{n-1}), \tag{42}$$

where

$$R_i = a_i \frac{(b_i w_n - b_n w_i)}{a_i b_i - a_n b_n} \quad \text{for} \quad i = 1,\ldots,n-1. \tag{43}$$

The time of switch, τ_1, of fire to the k^{th} target type is determined by the transcendental equation

$$w_n \cosh\sqrt{a_n b_n}\, \tau_1 + v\sqrt{\frac{b_n}{a_n}} \sinh \sqrt{a_n b_n}\, \tau_1 = \frac{a_k(b_k w_n - b_n w_k)}{a_k b_k - a_n b_n} . \tag{44}$$

This is seen to be a generalization of the case for two target types [11].

The general pattern of when and to which target types fire is shifted as we work backwards from the end of battle does not emerge until we have considered the second shift in target selection. Since this is dependent upon the evolution of target worth, we must further consider the backwards integration of the adjoint system of differential equations. From above, we have that

$$\varphi_i^*(\tau) = \delta_{ik} \quad \text{for} \quad \tau \in (\tau_1,\tau_2), \tag{45}$$

where τ_2 is the "backwards time" of the second switch in target selection. Giving consideration to (45), we see that for $\tau \in [\tau_1,\tau_2]$ we need only consider the following equations from the adjoint system (25)

$$\frac{dp_k}{d\tau} = -b_k p_{n+1} \quad \text{with} \quad p_k(\tau=\tau_1) = -W_k,$$

$$\frac{dp_{n+1}}{d\tau} = -a_k p_k \quad \text{with} \quad p_{n+1}(\tau=\tau_1) = V_k, \tag{46}$$

where

$$W_k = R_k = \frac{a_k(b_k w_n - b_n w_k)}{a_k b_k - a_n b_n}, \tag{47}$$

$$V_k = \sqrt{\frac{a_n}{b_n}(W_k^2 - w_n^2) + v^2}. \tag{48}$$

Equation (47) is merely (43) re-written with a chnage in notation. Equation (48) is readily deduced when we observe that according to (33) a "square law" relates the dual variables $p_n(\tau)$ and $p_{n+1}(\tau)$ for $0 \leq \tau \leq \tau_1$

$$a_n\{p_n^2(\tau) - w_n^2\} = b_n\{p_{n+1}^2(\tau) - v^2\}. \tag{49}$$

We further observe that all the dual variables may be expressed in terms of $p_k(\tau)$ (let $n = k$ in (26))

$$p_i(\tau) = \frac{b_i}{b_k}\{p_k(\tau) + w_k\} - w_i \quad \text{for} \quad i = 1,\ldots,n. \tag{50}$$

Again, the equations (46) are routinely solved to yield for $\tau \in [\tau_1, \tau_2]$

$$p_k(\tau) = -W_k \cosh\sqrt{a_k b_k}\,(\tau-\tau_1) - V_k\sqrt{\frac{b_k}{a_k}}\sinh\sqrt{a_k b_k}\,(\tau-\tau_1). \tag{51}$$

Subsequent arguments are now similar to those given for the first switch in tactics. Let j be the index of the target type

to which fire is shifted secondly in "backwards time." Then, it may be shown by similar arguments to above that necessary conditions for fire to be shifted to the j^{th} target type are that

$$a_j b_j > a_k b_k > a_n b_n, \tag{52}$$

and

$$\frac{b_j}{b_k} > \frac{w_j}{w_k}. \tag{53}$$

However, we gain more insight by considering (53) slightly differently, for it may be shown that

$$\frac{b_j}{w_j} > \frac{b_k}{w_k} > \frac{b_n}{w_n}. \tag{54}$$

It also seems appropriate to consider the military interpretation of the ratio $\frac{b_i}{w_i}$. We recall that

w_i = value per unit of X_i surviving at $t = T$,

b_i = kill rate per unit of X_i against Y.

Then

$$\frac{b_i}{w_i} = \frac{\text{kill rate per unit of } X_i}{\text{value per unit of } X_i \text{ survivors}}.$$

Thus, we see that as we progress backwards from the end of battle that fire is always shifted to target types with larger ratios of kill rate per unit of weapon system per unit value of survivors.

By a similar argument as for the first shift in fire using (42) and (43), it may be shown that the target to which fire is shifted secondly (working backwards from the end of battle) has index j determined by

$$S_j = \underset{\substack{S_i>0 \\ a_ib_i \; a_kb_k \\ i\neq k}}{\text{minimum}} (S_1,\ldots,S_n), \tag{55}$$

where

$$S_i = \frac{a_i(b_iw_k-b_kw_i)}{a_ib_i - a_kb_k} \qquad \text{for} \quad \underset{i\neq k}{i = 1,\ldots,n.} \tag{56}$$

The time of switch, τ_2, of fire to the j^{th} target type is determined by the transcendental equation

$$W_k \cosh\sqrt{a_kb_k}\,(\tau_2-\tau_1) + V_k\sqrt{\frac{b_k}{a_k}}\sinh\sqrt{a_kb_k}\,(\tau_2-\tau_1)$$

$$= \frac{a_j(b_jw_k-b_kw_j)}{a_jb_j - a_kb_k}. \tag{57}$$

Further shifts in fire follow the pattern established above.

Based on our above development, we now trace the course of battle forward in time. We assume that no force type is ever reduced to zero. (If this does happen, though, it may be shown that fire is merely shifted to the surviving target type of highest priority.) Then, we have that

(a) when $\tau_1 \geq T$

$$\phi_i^*(t) = \delta_{in} \quad \text{for} \quad 0 \leq t \leq T,$$

(b) when $\tau_2 \geq T > \tau_1$

$$\phi_i^*(t) = \delta_{ik} \quad \text{for} \quad 0 \leq t \leq T - \tau_1,$$
$$\phi_i^*(t) = \delta_{in} \quad \text{for} \quad T - \tau_1 \leq t \leq T - \tau_2,$$

(c) when $\tau_3 \geq T > \tau_2$

$$\phi_i^*(t) = \delta_{ij} \quad \text{for} \quad 0 \leq t \leq T - \tau_2,$$
$$\phi_i^*(t) = \delta_{ik} \quad \text{for} \quad T - \tau_2 \leq t \leq T - \tau_1,$$
$$\phi_i^*(t) = \delta_{in} \quad \text{for} \quad T - \tau_1 \leq t \leq T,$$

and so forth. τ_1 and τ_2 are determined by (44) and (57), respectively, and a similar expression exists for τ_3. The number of switches in target engagement depends upon the scheduled length of battle, T. As the battle progresses forward in time, fire is always shifted to a target type for which both $a_i b_i$ and $\frac{b_i}{w_i}$ are smaller than the target type previously engaged. If the battle is scheduled to last long enough, then during the initial stages of all fire is concentrated on the target type for which both the quantities $a_i b_i$ and $\frac{b_i}{w_i}$ are larger than any other target type.

4. <u>Several Target Types, Some Special Cases of Variable Attrition Rates</u>.

We consider the following prescribed duration battle:

$$\underset{\phi_i(t)}{\text{maximize}} \{vy(T) - \sum_{i=1}^{n} w_i x_i(T)\} \quad \text{with } T \text{ specified,}$$

$$\text{subject to: } \frac{dx_i}{dt} = -\phi_i a_i(t) y \quad \text{for } i = 1,\ldots,n,$$

$$\frac{dy}{dt} = -\sum_{i=1}^{n} b_i(t) x_i,$$

$$x_i, y \geq 0, \quad \phi_i \geq 0 \quad \text{for } i = 1,\ldots,n \quad \text{and} \quad \sum_{i=1}^{n} \phi_i = 1,$$

where all symbols are used in the same sense as above. We shall omit some details in the development of the solution to this problem, since the development is a direct synthesis of details from the two previous problems. In fact, an extensive elaboration on the nature of the optimal engagement rules will not be necessary, since we shall consider some special cases of variable attrition rates for which the solution is similar to the constant attrition-rate case considered in section 3.

In all cases, we assume that the X-force weapon systems have performance characteristics sich that

$$b_i(t) = k_{b_i} h(t) \quad \text{for } i = 1,\ldots,n. \tag{58}$$

As before, we first consider the special case when X-force types are valued in direct proportion to their kill rate against the Y forces at the end of battle

$$w_i = cb_i(t{=}T) = ck_{b_i} h(t{=}T). \tag{59}$$

The optimal allocation rule consequently takes a particularly simple form for $0 \leq t \leq T$

$$\phi_i^*(t) = \delta_{i,j(t)}, \tag{60}$$

where $j(t)$ is the index such that

$$a_j(t)b_j(t) = \text{maximum } (a_1(t)b_1(t),\ldots,a_n(t)b_n(t)).$$

Similar to our previous developments, the above solution is developed as follows. The adjoint system of differential equations is given by

$$\frac{dp_i}{dt} = b_i(t)p_{n+1} \quad \text{with} \quad p_i(t{=}T) = -w_i \quad \text{for} \quad i = 1,\ldots,n,$$

$$\frac{dp_{n+1}}{dt} = \sum_{i=1}^{n} \phi_i a_i(t)p_i \quad \text{with} \quad p_{n+1}(t{=}T) = v. \tag{61}$$

It is readily seen that assumption (58) leads to the result that

$$p_i(t) = \frac{k_{b_i}}{k_{b_n}} \{p_n(t) + w_n\} - w_i, \tag{62}$$

for $i = 1,\ldots,n$. It may be shown that the maximum principle leads to

$$\begin{aligned} &\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} \phi_i c_i(t) \\ &\text{subject to: } \sum_{i=1}^{n} \phi_i = 1, \\ &\qquad\qquad\quad \phi_i \geq 0, \end{aligned} \tag{63}$$

where

$$c_i(t) = \frac{(-p_n(t))}{k_{b_n}} a_i(t)k_{b_i} \left[1 + \frac{w_i}{(-p_n(t))}\left\{\frac{k_{b_n}}{k_{b_i}} - \frac{w_n}{w_i}\right\}\right]. \qquad (64)$$

The special assumption (59) that force types are valued in proportion to their kill rate leads to simplification of (64), namely

$$c_i(t) = \frac{(-p_n(t))}{k_{b_n}} a_i(t)k_{n_i} = \frac{(-p_n(t))}{b_n(t)} a_i(t)b_i(t),$$

whence our result (60), since it is readily shown that $p_n(t) < 0$.

We now consider the more complex case when enemy force types are not valued in proportion to their kill rates. Again, we consider a special case when relatively simple analytic results are still possible. Thus, we assume that

$$a_i(t) = k_{a_i} h(t) \quad \text{for} \quad i = 1,\ldots,n. \qquad (65)$$

In the case of a mobile attack against a static defensive position, assumptions (58) and (65) have the physical interpretation that all weapon systems and weapon system combinations have the same effective range and the same type of range dependency for their kill rate against any target type. As always, we develop the solution by working backwards from the end of battle; so it is convenient to introduce the "backwards time" $\tau = T - t$. Let n be the index of the target type fired on at the end of battle. Then

$$a_n(\tau=0)w_n = a_n(\tau=0)(-p_n(\tau=0))$$

$$= \text{maximum } (a_1(\tau=0)w_1, \ldots, a_n(\tau=0)w_n). \tag{66}$$

Then from (63) and (64) for $0 \leq \tau < \tau_1$, we have that

$$\phi_i^*(\tau) = \delta_{in},$$

where τ_1 is the "backwards time" of the first switch in target type at which the Y forces fire. As before, the above "bang-bang" optimal control leads to simplification of the adjoint system of differential equations, and it readily follows that

$$p_n(\tau) = -w_n \cosh \theta_n(\tau) - v\sqrt{\frac{k_{b_n}}{k_{a_n}}} \sinh \theta_n(\tau), \tag{67}$$

where

$$\theta_n(\tau) = \sqrt{k_{a_n} k_{b_n}} \int_0^\tau h(\tau)d\tau. \tag{68}$$

In deriving the above we have used the same results for variable coefficient differential equations used in section 2 and noted in [9].

Let k be the index of the target type to which fire is first shifted in "backwards time." Then for $\tau_1 < \tau < \tau_2$ we have that $\phi_i^*(\tau) = \delta_{ik}$. To determine the index k, we re-write (63) and (64) as

$$\underset{\phi_i}{\text{maximize}} \sum_{i=1}^{n} \phi_i e_i(\tau)$$

$$\text{subject to:} \quad \sum_{i=1}^{n} \phi_i = 1,$$

$$\phi_i \geq 0, \tag{69}$$

where $e_i(\tau)$ is given by

$$e_i(\tau) = a_i(t) w_i \left[1 + \frac{k_{b_i}}{w_i k_{b_n}} \left\{ (-p_n(\tau)) - w_n \right\} \right]. \tag{70}$$

The condition that $e_k(\tau) > e_n(\tau)$ for $\tau > \tau_1$ leads to

$$k_{a_k}(k_{b_k} w_n - k_{b_n} w_k) < (k_{a_k} k_{b_k} - k_{a_n} k_{b_n})(-p_n(\tau)), \tag{71}$$

which, in turn, yields that necessary conditions for fire to be shifted as we work backwards from the end to the k^{th} target type are

$$k_{a_k} k_{b_k} > k_{a_n} k_{b_n} \tag{72}$$

and

$$\frac{k_{b_k}}{w_k} > \frac{k_{b_n}}{w_n}. \tag{73}$$

Thus, it follows that the index k is determined by

$$R_k = \underset{\substack{R_i > 0 \\ k_{a_i} k_{b_i} > k_{a_n} k_{b_n}}}{\text{minimum}} (R_1, \ldots, R_{n-1}), \tag{74}$$

where

$$R_i = \frac{k_{a_i}(k_{b_i} w_n - k_{b_n} w_i)}{k_{a_i} k_{b_i} - k_{a_n} k_{b_n}} . \tag{75}$$

The "backwards time" of switch, τ_1, of fire to the k^{th} target type is seen to be given by the transcendental equation

$$w_n \cosh \theta_n(\tau_1) + v\sqrt{\frac{k_{b_n}}{k_{a_n}}} \sinh \theta_n(\tau_1) = R_k, \tag{76}$$

where $\theta_n(\tau)$ is given by (68). This is seen to be a generalization of corresponding results in the two previous sections.

Further details and results are similar to those presented in section 3. However, they differ in the same fashion as the earlier developments in this section differ from corresponding ones in section 3. We finally observe that the optimal target engagement policies are the same as those given at the very end of section 3. Changes in target priority are similarly determined (compare (42) and (43) with (74) and (75)). As noted elsewhere [9], [11], however, the time scale of battle has been transformed as evidenced by a comparison of values for τ_1 as determined by (44) with (76).

5. Comments.

In a companion paper [11], the interested reader can find an extensive discussion of the structure of the optimal target engagement policies for the problems whose solutions have been developed in this paper. We have contrasted there the structures of the optimal allocation policies for both these problems and also other tactical allocation problems. Here, we will make some further comments, however, about the optimal target engagement policies.

Let us first note, however, that for the problems considered in this paper the optimal control was always "bang-bang," i.e., an extreme point of the constrained control variable space. Thus, we saw that the optimal tactic was always to concentrate all fire on the appropriate target type. It may be shown that there are no singular extremals (see pp. 246-247 of [6]) on which the optimal control is an interior point of the control variable space. This happens, for example in the problem of section 2, because although the Hamiltonian is a linear function of the control variable, the coefficient of the control variable, i.e., $\frac{\partial H}{\partial \phi}$, cannot vanish over a finite interval of time. In other words, it is impossible that $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$.

Let us now consider the most general case of target selection in combat against heterogeneous forces, the problem of section 4. We did not include this problem in a companion paper for reasons

of brevity. For this problem, our basic assumption was that $b_i(t) = k_{b_i} h(t)$. Then, when survivors were valued in proportion to their effectiveness (i.e., kill rate against the Y forces), the optimal tactic took a remarkably simple form: always concentrate all fire on the target type for which $a_i(t)b_i(t)$ is the largest.

To obtain simple analytic results when X-force types were not valued in proportion to their kill rate, we further assumed that $a_i(t) = k_{a_i} h(t)$. Then as the battle developed forwards in time, the concentration of all fire on one target type was always shifted to a new target type for which both quantities $a_i(t)b_i(t)$ and $\frac{b_i(t)}{w_i}$ were smaller than those corresponding to the previously engaged target type.

In this paper we have attempted to show how some seemingly complex optimal control problems may readily yield relatively simple analytic solutions when special mathematical structures of the problem are exploited. We first pointed out such possibilities in a recent note [9] (where we inadvertently rediscovered some results apparently first observed by R. Isaacs). Although the solution to matrix differential equations (such as those encountered in sections 3 and 4) is, in general, messy at best when specific solutions are required, the occurrence of a "bang-bang" optimal control in these optimization problems reduced the complexity of the solution appreciably. We have purposely presented a sequence of problems to

show the reader how we set about to exploit such structural properties. Our personal opinion is that this could be done more frequently in applied areas like operations research where phenomena (here military operations) is to be studied.

6. Summary.

In this appendix we have presented the solution details for two problems discussed in a companion paper [11] and then considered a more general problem. We have shown how optimal controls may be synthesized in a simple fashion when special mathematical structures of the problems at hand are exploited. By considering a sequence of simplified, yet increasingly complex specific problems, we have hopefully laid the groundwork for studying more general and complex structures in the optimization of combat dynamics.

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Appendix D. Target Selection in Lanchester Combat: Linear-Law Attrition Process.

1. Introduction.

In this appendix we develop the solution to a simple problem of target selection in Lanchester combat against two enemy force types which undergo "linear-law" attrition. In addition to the Pontryagin maximum principle, the theory of singular extremals is required to solve this problem. Our major contribution is to show how to synthesize the optimal target selection policies from the basic optimality conditions. This solution synthesis methodology is applicable to more general dynamic (tactical) allocation problems. For constant attrition rates we show that whether or not changes can occur in target priorities depends solely on how survivors are valued and is independent of the type of attrition process.

In Appendix B we have presented some elements of a mathematical theory of target selection in dynamic combat situations. We did this through the examination of the structure of the optimal allocation policies for some tactical situations described

by Lanchester-type equations of warfare. The purpose of this previous appendix was to contrast the structures of the optimal allocation policies for various scenarios without cluttering the discourse with the mathematical details of solution development. In the present appendix we develop results for a prescribed duration battle in which enemy target types undergo a "linear-law" attrition process (see section 3 below). We had previously [8] just stated these results without justification.

The problem under study is solved by the mathematical theory of optimal control. Its solution, however, requires more than the well-known Pontryagin maximum principle [7]: the theory of singular extremals (see Chapter 8 of [2]) must be used to solve it. A brief discussion of the required theory of singular extremals is included in this appendix. By an extremal we mean a battle trajectory on which the necessary conditions for optimality are almost everywhere in time satisfied.

The major contribution of this appendix is to show how to synthesize the optimal control in combat against target types which undergo a "linear-law" attrition process. In this case, singular

subarcs (see section 2. below) may be present in the battle trajectory. By the synthesis of optimal control, we mean the explicit determination of the time history of the optimal control from initial to terminal time as a function of the initial state of the system. There is no general method for the synthesis of optimal controls in singular problems [6]; each class of problems possesses its own peculiarities. Hence, an understanding of how to synthesize the optimal control in this elementary problem is particularly important, since it provides insight for more complex extensions that we have considered in our subsequent researches.

The body of this appendix is organized in the following fashion. First, we review that part of the theory of singular extremals which is required for the solution of the problem under study. Next, we present our model and develop the basic necessary conditions of optimality. Then, we show how to synthesize the solution to our problem. This is done for the two cases of import. Finally, we make some comments about the structure of the optimal target selection policies and extensions.

2. The Theory of Singular Extremals.

In an optimal control problem, the maximum principle may fail to determine an optimal trajectory, since the maximization of the Hamiltonian may not lead to a well-defined expression for optimal control [5], [4] (also see Chapter 8 of [2]). Singular solutions usually occur when the Hamiltonian is a linear function of the

control variables. However, all problems for which the Hamiltonian is a linear function of the control variables do not have singular subarcs in their solution.

The problem that we shall consider has one control variable, and it appears linearly in the Hamiltonian. By a singular subarc we denote that part of an optimal trajectory on which the maximum principle cannot be used to determine the control because the coefficient of the control variable in the Hamiltonian is zero (see pp. 226-227 of [4]). Then the term "singular soltuion" will be used to refer to any optimal trajectory which contains one or more singular subarcs.

To elaborate further, when the Hamiltonian H is a linear function of the control variable ϕ, then if $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time (or, another way to say this, the coefficient of ϕ vanishes identically for a finite interval of time), then the maximum principle does not determine the control. Observe that in this case all feasible values of ϕ maximize the Hamiltonian. When this happens we determine the singular control by requiring that we remain on the singular subarc, i.e. $\frac{\partial H}{\partial \phi}$ remains zero. If $\frac{\partial H}{\partial \phi}$ is to be identically equal to zero for a finite interval of time, then all its derivatives with respect to time must also be equal to zero. We determine the singular control, which keeps the system on the singular subarc, by considering as many of the time derivatives of $\frac{\partial H}{\partial \phi}$ as are required for the control variable ϕ to appear explicitly

so that it may be determined from an algebraic equation. Thus, in general we consider

$$0 = \frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = \ldots \tag{1}$$

For the problem at hand, the equation $\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = 0$ and the canonical equations (i.e. both state and adjoint system) lead to an explicit expression for the singular control.

We must further check to make sure that we can get a maximum return (in the case when we wish to maximize the criterion functional) from use of the candidate singular subarc. The following condition (generalized Legendre-Clebsch condition) is necessary for a singular subarc to yield a maximum return

$$(-1)^k \frac{\partial}{\partial \phi}\left\{\frac{d^{2k}}{dt^{2k}}\left(\frac{\partial H}{\partial \phi}\right)\right\} \leq 0. \tag{2}$$

It is obtained by examining the negative semidefiniteness of the second variation for a special class of explicitly defined control variations [6]. For the problem at hand, it suffices to consider the generalized Legendre-Clebsch condition with k = 1. Recently, Jacobson [3] has discovered a new necessary condition for optimality on singular subarcs. This condition is not readily checked, however, for the problem at hand, since the details of application are extremely messy.

3. The Model and Development of Basic Optimality Conditions.

We consider the following prescribed duration battle

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 x_1 y$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2,$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1, \tag{3}$$

where

p, q and r are values placed on surviving force types,

x_1, x_2 and y are average force strengths,

a_1, a_2, b_1 and b_2 are constant attrition rates,

(observe that the a's and b's are different in nature),

ϕ is the fraction of Y-fire directed X_1.

In previous papers [8], [9] we have described the basic scenario under consideration and also the circumstances which lead to a "linear-law" attrition process. As before [8], we refer to attrition as being a "linear-law" process when the casualty rate is proportional to the product of the number of enemy firers and remaining targets. We have also discussed at length the structure of the optimal target selection policies and its implications for military tactics previously [8].

We now develop the basic optimality conditions which hold on extremals. The Hamiltonian for the above problem is given by

$$H(t,x_i,p_i,\phi) = (-p_1a_1x_1y + p_2a_2x_2y)\phi + \{-p_2a_2x_2y - p_3(b_1x_1 + b_2x_2)\}, \quad (4)$$

where $p_i(t)$ for $i = 1,2,3$ are the dual variables corresponding to the state variables $x_1,x_2,x_3 = y$ (see [8], [9] for a discussion of the military significance of these variables). The maximum principle leads to the following non-singular optimal control (there is only one extremal)

$$\phi^*(t) = \begin{cases} 0 & \text{for } p_2a_2x_2 < p_1a_1x_1, \\ 1 & \text{for } p_2a_2x_2 > p_1a_1x_1. \end{cases} \quad (5)$$

The adjoint system of differential equations for the dual variables is given by

$$\frac{dp_1}{dt} = \phi a_1yp_1 + b_1p_3 \qquad \text{with } p_1(t = T) = -p,$$

$$\frac{dp_2}{dt} = (1 - \phi)a_2yp_2 + b_2p_3 \qquad \text{with } p_2(t = T) = -q,$$

$$\frac{dp_3}{dt} = \phi a_1x_1p_1 + (1 - \phi)a_2x_2p_2 \qquad \text{with } p_3(t = T) = r. \quad (6)$$

Conditions (1) for a singular subarc yield for the problem at hand

$$p_1a_1x_1 = p_2a_2x_2, \quad (7)$$

and

$$a_1b_1x_1 = a_2b_2x_2. \quad (8)$$

Differentiation of (8) and combination with the state equations (3) yields the singular control required to remain on a singular subarc

$$\phi^* = \frac{a_2}{a_1 + a_2}. \tag{9}$$

Checking the generalized Legendre-Clebsch condition for this singular subarc, we find after a rather laborious computation which requires use of both (3) and (6) that

$$\frac{\partial}{\partial\phi}\{\frac{d^2}{dt^2}(\frac{\partial H}{\partial\phi})\} = y^2 p_3(t)\{(a_1)^2 b_1 x_1 + (a_2)^2 b_2 x_2\} > 0,$$

since it is readily shown that $p_3(t) > 0$ for all t. Hence, the necessary condition is met for the singular path to be optimal.

In synthesizing the optimal course of battle (backwards from the end of the prescribed duration battle) it is convenient to introduce

$$v(t) = -a_1 p_1 x_1 + a_2 p_2 x_2. \tag{10}$$

By (5) and (9) the optimal control may be expressed in terms of v(t) as

$$\phi^*(t) = \begin{cases} 1 & \text{for } v(t) > 0, \\ \dfrac{a_2}{a_1 + a_2} & \text{for } v(t) = 0, \\ 0 & \text{for } v(t) < 0. \end{cases} \tag{11}$$

We recall that (8) must also hold on a singular subarc.

Since we develop the solution to this problem by working backwards from the end t = T, it is convenient to introduce the

"backwards time" variable τ defined by $\tau = T - t$. Observing that $\frac{d}{dt} = -\frac{d}{d\tau}$ and using both the state equations (3) and the adjoint system (6), we obtain from differentiation of (10) that

$$\frac{dv}{d\tau} = (a_1 b_1 x_1 - a_2 b_2 x_2) p_3. \tag{12}$$

Thus, we see that on a singular subarc on which $v(\tau) = 0$ we also have that $\frac{dv}{d\tau} = 0$. Also, it is sometimes convenient to write (10) as

$$v(\tau) = -\left(\frac{p_2(\tau)}{b_2}\right)\left[\frac{\left(\frac{p_1(\tau)}{p_2(\tau)}\right)}{\left(\frac{b_1}{b_2}\right)}(a_1 b_1 x_1) - a_2 b_2 x_2\right]. \tag{13}$$

At the end of battle $t = T$, we have

$$v(\tau = 0) = a_1 p x_1(t = T) - a_2 q x_2(t = T). \tag{14}$$

Taking (13) into consideration, we see that a point on the singular "surface" $a_1 b_1 x_1 = a_2 b_2 x_2$ yields a positive, zero, or negative value for $v(\tau)$ at $\tau = 0$ depending upon whether $\frac{p}{q}$ is greater than, equal to, or less than $\frac{b_1}{b_2}$. Hence, by (11) a battle trajectory which has reached the singular surface can, in general, only remain on it at the end of battle when $\frac{p}{q} = \frac{b_1}{b_2}$. Thus, in synthesizing optimal trajectories we must consider three cases.

Case (a) $\frac{p}{q} = \frac{b_1}{b_2}$,

Case (b) $\frac{p}{q} > \frac{b_1}{b_2}$,

Case (c) $\frac{p}{q} < \frac{b_1}{b_2}$.

The solution for Cases (a) and (b) has been described by us in a previous paper [8].

If we were to plot in Figure 1 the line L' defined by $a_1px_1 = a_2qx_2$, then it would appear above, on, or below the line L defined by $a_1b_1x_1 = a_2b_2x_2$ depending on whether $\frac{p}{q}$ were greater than, equal to, or less than $\frac{b_1}{b_2}$. This is evident from considering the slopes of these two lines.

$$\left(\frac{dx_2}{dx_1}\right)_L = \frac{a_1b_1}{a_2b_2}, \qquad \left(\frac{dx_2}{dx_1}\right)_{L'} = \frac{a_1p}{a_2q},$$

since, for example,

$$\frac{p}{q} > \frac{b_1}{b_2} \text{ implies that } \left(\frac{dx_2}{dx_1}\right)_{L'} > \left(\frac{dx_2}{dx_1}\right)_L .$$

The significance of the line L' and its relationship to the line L is as follows. The battle is divided into two time phases: Phase I for $0 \leq t \leq t_1 = T - \tau_1$ and Phase II for $T - \tau_1 = t_1 \leq t \leq T$. During Phase I the optimal target engagement policy at a point in time is determined by the location of the point on the battle trajectory with respect to the line L, which is also the singular "surfaces." Above L, $\phi^*(t) = 0$; while below L, $\phi^*(t) = 1$. When a battle trajectory reaches L, it remains on the singular surface through use of the singular control $\phi^* = \frac{a_2}{a_1 + a_2}$. During Phase II

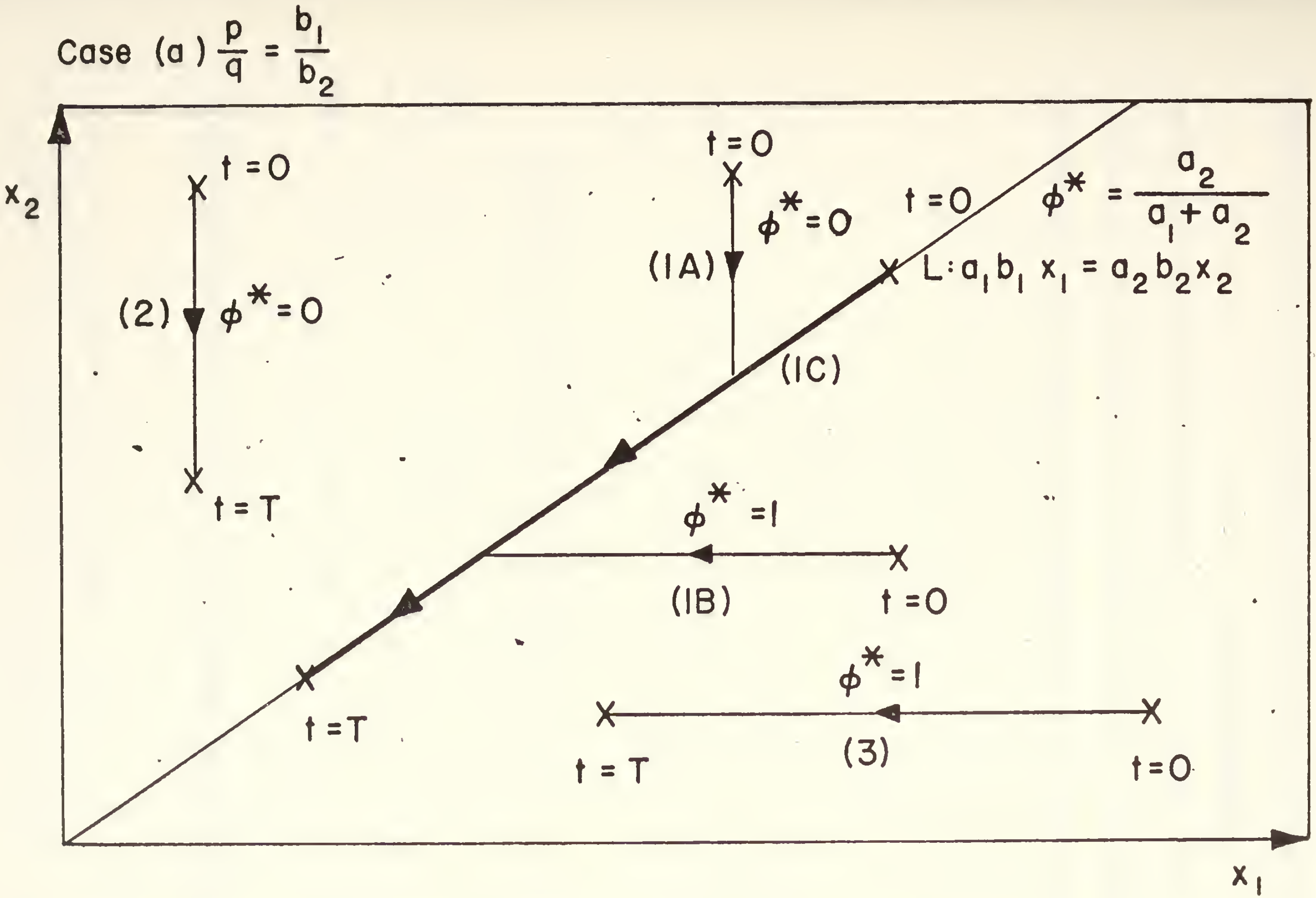

Figure 1. Optimal allocation for linear-law attrition process - Survivors valued in direct proportion to kill rates.

the optimal target engagement policy is to use $\phi^*(t) = 1$ below L'. It may be shown that it is impossible for a battle trajectory to cross L' during Phase II.

The above results will be developed in the next two sections on the synthesis of optimal control. The following relationships readily follow from previously developed results and are required to establish the results of the above paragraph

$$v(\tau=0) \begin{cases} > 0 & \text{below } L', \\ < 0 & \text{above } L', \end{cases} \tag{15}$$

so that

$$\phi^*(t=T) = \begin{cases} 1 & \text{for } P(T) \text{ below } L', \\ 0 & \text{for } P(T) \text{ above } L', \end{cases} \tag{16}$$

where $P(t=T) = (x_1(t=T), x_2(t=T))$. We also note that by (12)

$$\frac{dv}{d\tau}(\tau) \begin{cases} > 0 & \text{below } L, \\ = 0 & \text{on } L, \\ < 0 & \text{above } L. \end{cases} \tag{17}$$

4. <u>Solution Synthesis When Survivors Valued in Proportion to Kill Rates</u>.

For Case (a): $\frac{p}{q} = \frac{b_1}{b_2}$, optimal battle trajectories are shown in Figure 1. Above the line L with equation $a_1b_1x = a_2b_2x_2$ the

optimal control is to use $\phi^*(t) = 0$ until this line is encountered. When a trajectory reaches L, the singular control $\phi^* = \frac{a_2}{a_1 + a_2}$ (which keeps the trajectory on L) is used until the end of battle at $t = T$. Below L, $\phi^*(t) = 1$ is used in a similar fashion. To establish these results, we work backwards from each possible type of end point of battle.

At the end of battle $\tau = 0$ equation (13) reduces to

$$v(=0) = \left(\frac{q}{b_2}\right)[a_1b_1x_1(t{=}T) - a_2b_2x_2(t{=}T)], \tag{18}$$

since we have assumed $\frac{p}{q} = \frac{b_1}{b_2}$. By (18) we see that there are three cases to consider depending on the sign of the term in square brackets.

<u>Case (1)</u> $a_1b_1x_1(t{=}T) = a_2b_2x_2(t{=}T)$

This corresponds to when the system ends up on the singular subarc. In this case $\phi^*(t{=}T) = a_2/(a_1+a_2)$, and for $0 \leq \tau \leq \tau_1$ = the "backwards time" of the first switch, we use the singular control $\phi^*(\tau) = a_2/(a_1+a_2)$. Let us note that use of the singular control for $0 \leq \tau \leq \tau_1$ results in $\frac{dv}{d\tau} = 0$ so that $v(\tau) = v(\tau{=}0) + \int_0^\tau \frac{dv}{d\tau}\, d\tau = 0$. At $t_1 = T - \tau_1$ we switch control, since $x_1(t_1) = x_1^\circ$ or $x_2(t_1) = x_2^\circ$. This yields three further subcases.

<u>Subcase (1A)</u> $a_1b_1x_1^\circ < a_2b_2x_2^\circ$

At $t = t_1 > 0$ we have that $a_1b_1x_1^\circ = a_2b_2x_2(t_1) < a_2b_2x_2^\circ$ so

that we cannot destroy anymore x_1. Then we use $\phi^*(\tau) = 0$ for $\tau_1 \leq \tau \leq T$. This is consistent since $v(\tau=\tau_1) = 0$ and

$$\frac{dv}{d\tau} = p_3(a_1 b_1 x_1^\circ - a_2 b_2 x_2) < 0 \quad \text{for} \quad \tau_1 \leq \tau \leq T.$$

(Observe that for $\tau_1 < \tau \leq T$, $\frac{dx_2}{d\tau} = a_2 x_2 y$ so that $x_2(\tau) > x_2(\tau_1)$.) This implies that $v(\tau) < 0$, and hence $\phi^*(t) = 0$ for $0 \leq t \leq t_1 = T - \tau_1$.

<u>Subcase (1B)</u> $\quad a_1 b_1 x_1^\circ > a_2 b_2 x_2^\circ$

A similar argument readily yields that $\phi^*(t) = 1$ for $0 \leq t \leq t_1$.

<u>Subcase (1C)</u> $\quad a_1 b_1 x_1^\circ = a_2 b_2 x_2^\circ$

We use $\phi^*(t) = a_2/(a_1+a_2)$ from the beginning.

<u>Case (2)</u> $\quad a_1 b_1 x_1(t=T) < a_2 b_2 x_2(t=T)$

Since $v(\tau=0) = \left(\frac{q}{x_2}\right)[a_1 b_1 x_1 - a_2 b_2 x_2] < 0$, at the end of battle we have $\phi^*(t=T) = 0$. Hence, for $0 \leq \tau \leq \tau_1$ = the "backwards time" of the first switch, we use $\phi^*(\tau) = 0$. We work backwards from the end. Since we are above the line L, $\frac{dv}{d\tau} = p_3(a_1 b_1 x_1 - a_2 b_2 x_2) < 0$. Hence, $v(\tau) < 0$ for all $\tau \in [0,T]$, and we never do switch. Thus, we have that $\phi^*(t) = 0$ for $0 \leq t \leq T$.

<u>Case (3)</u> $\quad a_1 b_1 x_1(t=T) > a_2 b_2 x_2(t=T)$

A similar argument to that used for Case (2) readily yields that $\phi^*(t) = 1$ for $0 \leq t \leq T$.

The above cases are shown in Figure 1. It should be noted that the above development depends upon the easily proven fact that $p_3(t) > 0$ for all t. It should further be noted that, in general, trajectories (1A), (1B) and (1C) will not all terminate in the same point as shown in Figure 1, which was drawn this way for simplicity.

5. Solution Synthesis when Survivors Not Valued in Proportion to Kill Rates.

We now consider Case (b): $\frac{p}{q} > \frac{b_1}{b_2}$. Again, we work backwards from each possible type of end point of battle. There are two cases to be considered.

Case (1): never on singular subarc for finite interval of time.

Again there are two subcases to consider, depending upon whether the system winds up above or below L.

Subcase (1a) $\quad a_1b_1x_1(t=T) \geq a_2b_2x_2(t=T)$

Since

$$v(\tau) = a_1b_1x_1\left(\frac{-p_2}{b_2}\right)\left[\frac{(p_1/p_2)}{(b_1/b_2)} - \frac{a_2b_2x_2}{a_1b_1x_1}\right],$$

we see that $v(\tau=0) > 0$ and hence by (11) $\phi^*(=T) = 1$. Hence for $0 \leq \tau \leq \tau_1$ = the "backwards time" of the first switch, we use $\phi^*(\tau) = 1$. We work backwards from the end using this control. Since

$$\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2) > 0$$

when we are below L and we stay there by using $\phi^*(t) = 1$, we have that $v(\tau) > 0$ for all $\tau \in [0,T]$, and hence we never switch. Thus, $\phi^*(t) = 1$ for $0 \leq t \leq T$.

<u>Subcase (1b)</u> $\quad a_1b_1x_1(t=T) < a_2b_2x_2(t=T)$

Again there are two further subcases to consider, depending upon whether the system winds up above or below L'.

<u>Subcase (1bI)</u> $\quad a_1b_1x_1(t=T) < a_2b_2x_2(t=T)$ <u>and</u>

$$a_1px_1(t=T) < a_2qx_2(t=T)$$

In this case we wind up above L' and hence by (16) $\phi^*(t=T) = 0$. Since we are above L, $\frac{dv}{d\tau} < 0$ for all τ by (17). Combining this with (15), it readily follows that $v(\tau) < 0$ for all $\tau \in [0,T]$. Thus, $\phi^*(t) = 0$ for $0 \leq t \leq T$.

<u>Subcase (1bII)</u> $\quad a_1b_1x_1(t=T) < a_2b_2x_2(t=T)$ <u>and</u>

$$a_1px_1(t=T) > a_2qx_2(t=T)$$

In this case we wind up below L' at the end. By (15) and (16), we have that $v(\tau=0) > 0$ and $\phi^*(\tau=0) = 1$. We work backwards from the end. Since we are above L, $\frac{dv}{d\tau} < 0$ by (17) while we remain above L. Thus $v(\tau)$ decreases as τ increases. There are two further subcases depending upon whether $v(\tau)$ decreases to zero before the line L is encountered. Let τ_1 be such that $v(\tau_1) = 0$. If L has not been reached at τ_1, then $v(\tau)$ for $\tau > \tau_1$ is negative and $\phi^*(\tau) = 0$ for $\tau_1 \leq \tau \leq T$. It is also possible to just reach L

when $v(\tau_1) = 0$. In this case (assuming that we don't remain on the singular subarc) $v(\tau) > 0$ for $\tau > \tau_1$, since we pass below L and then $\frac{dv}{d\tau} > 0$.

Case (2): on singular subarc for finite interval of time

Considering (15) and (17), it is readily seen that this can only happen when $a_1 b_1 x_1(t=T) < a_2 b_2 x_2(t=T)$ and $a_1 p x_1(t=T) > a_2 q x_2(t=T)$. As usual, we work backwards from the end of battle. By previous arguments it is readily seen that we use $\phi^*(\tau) = 1$ for $0 \leq \tau \leq \tau_1$, and at $\tau = \tau_1$ we must have $a_1 b_1 x_1(\tau_1) = a_2 b_2 x_2(\tau_1)$. We use the singular control $\phi^*(\tau) = a_2/(a_1+a_2)$ for $\tau_1 \leq \tau \leq \tau_2$. There are three further subcases.

(1) $x_1(\tau_2) = x_1^\circ \quad , \quad x_2(\tau_2) < x_2^\circ$,

(2) $x_1(\tau_2) < x_1^\circ \quad , \quad x_2(\tau_2) = x_2^\circ$,

(3) $x_1(\tau_2) = x_1^\circ \quad , \quad x_2(\tau_2) = x_2^\circ$.

We omit the trivial discussion of these cases.

Thus we see from the above that there are six possible cases for the history of combatant force strengths in this prescribed duration battle:

(1) started below L and never reached L,

(2) always above L',

(3) started above L' and end up above L but below L' without ever reaching L,

(4) end up above L but started below L and did not remain on L for finite interval of time,

(5) started above (or on) L and were on L for finite interval of time,

(6) started below L and were on L for finite interval of time.

These six cases are shown in Figure 2. Case (c): $\frac{p}{q} < \frac{b_1}{b_2}$ is similar to Case (b).

6. Comments.

Elsewhere [8] we have contrasted the structure of the optimal target engagement policies in Lanchester combat when the engaged target types undergo a "linear-law" attrition process with that for other tactical scenarios. An important question to be answered in such studies is whether target priorities change over time. We have discovered that for the scenarios which we have so far studied the answer to this question is determined solely by whether or not surviving target types are valued in direct proportion to their kill-rate capabilities. For the case of constant attrition rates, changes in target priorities over time can only occur when survivors are valued in excess of their kill-rate capabilities. This is true when the engaged target types are undergoing either a "linear-law" attrition process or a "square-law" one (see [9] for a discussion of the "square-law" case).

We now discuss how the above principle applies to the problem at hand. When a linear utility is assigned to enemy survivors at the end of battle in direct proportion to their kill-rate (effectiveness)

Case (b) $\frac{p}{q} > \frac{b_1}{b_2}$

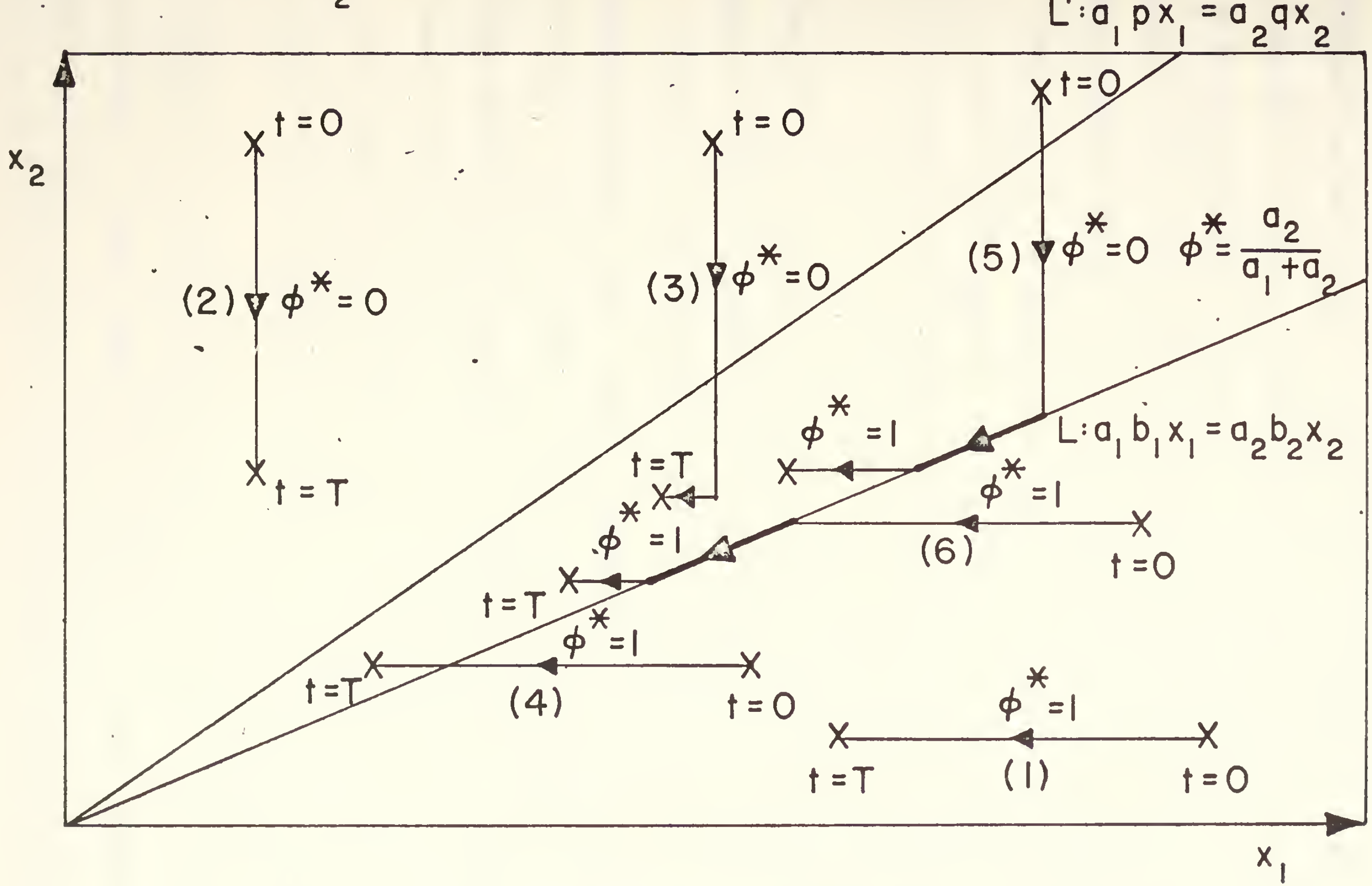

Figure 2. Optimal allocation for linear-law attrition process - Survivors not valued in direct proportion to kill rates.

against friendly forces, then the optimal target selection policy depends only upon the location of the battle trajectory with respect to the singular "surface" L (see Figure 1). Thus, target priorities don't change over time (they can become equal, however). When one target type is assigned utility in excess of its effectiveness (i.e., $p/q > b_1/b_2$), then at time t_1 there will be a switch from tactics being determined by the location of the battle trajectory with respect to the singular "surface" L to being determined by location with respect to the line L' (see Figure 2). It may be shown that t_1 depends on the particular battle trajectory under consideration and no trajectory can "penetrate" L'.

The methodology for solution synthesis developed in this paper is applicable to more complex tactical situations of greater military significance. Our work here lays the foundations for the study of the optimal allocation of supporting weapon systems (e.g., artillery, tactical air support, etc.) against "area targets" (e.g., troop concentrations). Typical questions of interest to be answered are, "Considering several infantry companies individually engaging enemy units of like size, what is the 'best' utilization of supporting artillery fires?" or, "What is the 'best' utilization of Naval fire support in amphibious assaults?"

In a previous paper [8], we have pointed out that the structure of the optimal allocation policies in Lanchester combat is basically determined by whether there are constant attrition returns over time

per unit of weapon system employed or diminishing returns. In the present paper we have studied target selection with diminishing returns over time, i.e., "linear-law" attrition process. It should be noted that there is a problem in the literature with similar solution structures, the continuous version of Bellman's stochastic gold-mining process (see pp. 222-233 of [1]). When there are diminishing returns over time from the use of a device subject to breakdown, then the problem of maximizing the return from use of one device in either of two potential locations has a similar structure to the optimization problem in Lanchester combat studied here. The interested reader should compare the solution as shown in our Figure 1 with that of Theorem 1 on p. 231 of [1] and also our Figure 2 with Figure 4 on p. 232 of [1]. When the stochastic gold-mining problem is re-examined by modern optimal control theory, new insights are gained into the operation of maximizing the return from a resource subject to breakdown or loss, and we shall discuss this in the future.

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Appendix E. The Theory of State Variable Inequality Constraints.

1. Introduction.

Optimal control problems involving inequality constraints on a function of the state variables with no explicit dependence on the control variables have been treated only in recent years. The pioneering work of the Russian R. Gamkrelidze in 1959 (see Chapter VI in [22]) was followed in the U.S.A. by that of L. Berkovitz [3], S. Dreyfus [12], and Bryson, Denham, and Dreyfus [9]. Berkovitz and Dreyfus [5] subsequently have shown the equivalence of the results of Gamkrelidze and Berkovitz with those of Dreyfus. McIntyre and Paiewonsky [20] have written an excellent survey article on the theory of state variable inequality constraints (SVIC) and summarize theoretical results through 1966.

Furthermore, in 1967 McIntyre and Paiewonsky [20] remarked that "the optimal control problem with state space constraints does not appear to be well understood." Our own experience bears this out. We have recently pointed out [27] that certain theoretical results cited in the literature are inadequate by themselves to determine optimal trajectories in a certain non-autonomous class of problems but must be supplemented by a result due to Gambrelidze [22].

Even more incredible is the fact that for two-sided variational problems (i.e. two-person zero-sum deterministic differential games) we are not aware of any adequate treatment of inequality

constraints of functions of the state variables only (the control variables not explicitly appearing). Although R. Isaacs does consider such a problem (the "War of Attrition and Attack" [16]), we will show that his solution is unsupported by proper analysis, and we will indicate a similar problem for which his approach fails to yield optimal strategies. L. Berkovitz does not consider problems with state variable inequality constraints (SVIC) in [4]; and although A. Friedman claims to treat such problems [13] (see also Chapter 6 in [14]), he fails to develop the appropriate necessary conditions of optimality which must hold when such a constraint is active (i.e. on a constrained subarc).

In this appendix we shall review some necessary conditions of optimality which must hold on a constrained subarc in a one-sided problem with a SVIC. Following the approach of L. Berkovitz [4], we may readily extend these conditions to differential games. We propose this to ONR as a future research task.

2. Basic Approaches.

There are two basic approaches to treating a SVIC in an optimal control problem. The first and more direct approach consists in adjoining the state-variable constraint directly to the criterion functional with an additional Lagrange multiplier. This approach has been considered by Chang [10] and Speyer and Bryson [23] (see also [18]). We will not consider this method of direct adjoining further in this appendix.

The second method consists in adjoining the time derivative of the state-variable constraint to the criterion functional with an additional Lagrange multiplier. We say [9] that the problem has a k^{th} order state variable inequality constraint (SVIC) when the first time (total) derivative of the state-variable constraint which explicitly contains the control variable(s) is the k^{th}. This is the approach originally considered by Gamkrelidze [22] for a first order SVIC and further extended to higher order SVIC by Bryson et al. [9]. Furthermore, this is the approach that we shall take in this appendix.

3. Necessary Conditions of Optimality When Time Derivative of Constraint Is Adjoined to Return Functional.

It suffices to consider a problem with a single control variable and a single inequality constraint on the state variables. Without loss of generality we may assume that the problem's planning horizon is for a fixed period of time. We further consider a (possibly different) inequality constraint on the state variables at the end of the planning horizon.

Let us therefore consider the problem

$$\underset{u(t)}{\text{maximize}} \quad \lambda(x_i(T)) + \int_0^T L(t,x_i(t),u(t))dt, \quad \text{with } T \text{ specified,}$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = f_i(t,x_i,u) \qquad \text{for} \quad i = 1,\ldots,n, \tag{1}$$

$u(t) \in U$ (u(t) is scalar control variable suitably restricted),

$C(t,x_i) \leq 0$ (scalar inequality constraint on state variables),

$\psi(x_i(T)) \leq 0$ (scalar inequality constraint on state variables at terminal time),

where we assume that all functions are smooth enough to insure the existence of all partial derivatives required in the following analysis. We shall consider separately the cases of a first order SVIC and a k^{th} $(k > 1)$ order SVIC.

a. First Order SVIC.

Adjoining the first time derivative of the state-variable constraint to the return functional, we have that the Hamiltonian is given by [8], [9], [11], [22]

$$H(t,x_i,p_i,u) = L(t,x_i,u) + \sum_{i=1}^{n} p_i f_i(t,x_i,u) - \mu(t)\,\frac{dC}{dt}\,, \tag{2}$$

where

$$\mu(t) \begin{cases} = 0 & \text{for} \quad C(t,x_i) < 0\,, \\ \geq 0 & \text{for} \quad C(t,x_i) = 0\,. \end{cases}$$

We may, of course, write (2) in expanded form as

$$H(t,x_i,p_i,u) = L(t,x_i,u) + \sum_{i=1}^{n} p_i f_i(t,x_i,u)$$

$$-\mu(t)\{\frac{\partial C}{\partial t} + \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} f_i(t,x_i,u)\} \; . \tag{3}$$

The adjoint equations for the dual variables are given by

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial x_i} = - \frac{\partial L}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} p_j + \mu(t) \frac{\partial}{\partial x_i} \left(\frac{dC}{dt}\right) \; , \tag{4}$$

where $p_i(t)$ is the dual variable corresponding to the state variable x_i. Let us assume (as is invariably) that all state variables are specified at $t = 0$, i.e. we are given $x_i(t=0) = x_i$ for $i = 1,\ldots,n$. Hence, $p_i(t=0)$ is unspecified, and we have the following boundary conditions for the dual variables at $t = T$

$$p_i(t=T) = \frac{\partial \lambda(T)}{\partial x_i} - \nu \frac{\partial \psi}{\partial x_i} (T) \; , \tag{5}$$

for $i = 1,\ldots,n$ where

$$\nu \begin{cases} = 0 & \text{for } \psi(x_i(T)) < 0 \; , \\ \geq 0 & \text{for } \psi(x_i(T)) = 0 \; . \end{cases}$$

Conditions (5) were apparently first observed by Funk and Gilbert

[15]. (A special case of conditions (5) was earlier observed by Arrow and Kurz [2] although their derivation is only heuristic.) Its proof readily follows along the lines of L. Neustadt's development in [21]. (It should be noted that (5) has been stated for the prescribed duration problem. It need not hold in other cases (for example, in the case when the planning horizon ends when a target set (exclusive of time) is reached).)

On a constrained subarc on which $C(t,x_i(t)) = 0$ for $0 \le t_1 \le t \le t_2 \le T$ we have that

$$\frac{dC}{dt}(t,x_i(t)) = 0 \qquad \text{for} \quad t_1 < t < t_2 ,$$

and hence the control is determined by

$$\frac{\partial C}{\partial t} + \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} f_i(t,x_i,u^*) = 0 . \tag{6}$$

The Lagrange multiplier $\mu(t)$ is determined by

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial L}{\partial u} + \sum_{i=1}^{n} p_i \frac{\partial f_i}{\partial u} - \mu(t) \frac{\partial}{\partial u}\left(\frac{dC}{dt}\right) . \tag{7}$$

Bryson, Denham, and Dreyfus [9] derived that on a constrained subarc (of finite length in time) a necessary condition of optimality is

$$\mu(t) \ge 0 \qquad \text{for} \quad t_1 \le t \le t_2 . \tag{8}$$

For this problem (1) Gamkrelidze (see Chapter VI in [22]) (who did not observe (8)) developed the following additional necessary condition

$$\dot{\mu}(t) = \frac{d\mu}{dt} \leq 0 \qquad \text{for} \quad t_1 \leq t \leq t_2 \, . \tag{9}$$

We have recently pointed out [27] that many workers [1], [8], [9], [11] have overlooked the importance of (9). It should be noted that results corresponding to (8) and (9) have not been developed for differential games (see, in particular, [4], [14], [16]).

Furthermore, there are corner conditions (for a discussion of corner conditions within the framework of the classical calculus of variations see p. 38 of [6] or p. 367 and p. 571 of [7]) that must be satisfied upon entering to and exiting from a constrained subarc. (These important conditions are never mentioned in [1] or [2].) Let t_1 denote the entry time to the constrained subarc and t_2 denote the exit time. This situation is depicted in Figure 1.

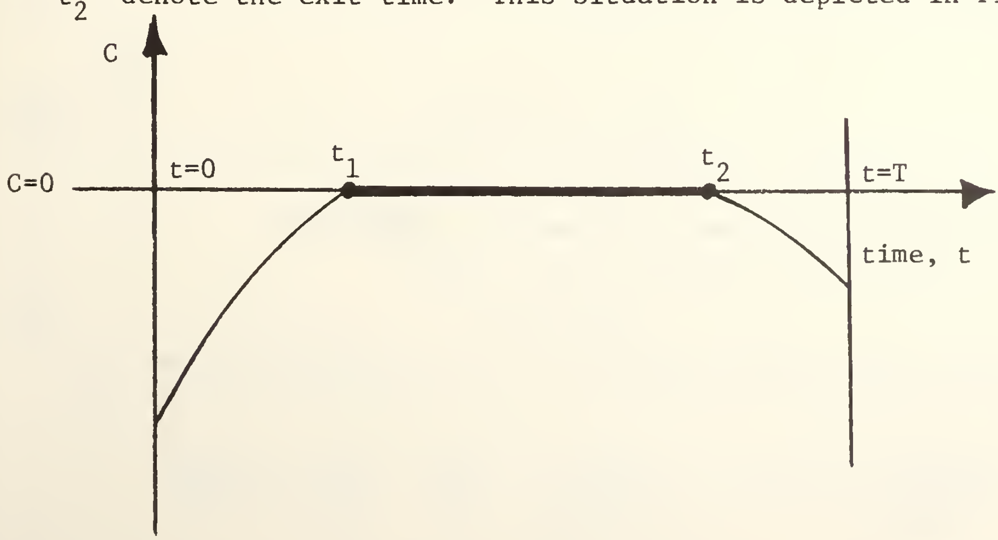

Figure 1. Entry to and Exit from Constrained Subarc.

Furthermore, let t_1^- denote a left-hand limit, i.e. $t_1^- = \lim\limits_{\substack{t \to t_1 \\ t \le t_1}} t$.

It has been established [9] that the total discontinuity in the adjoint variables can be taken at the entry corner (i.e. $t = t_1$). Thus, we have <u>at entrance corner</u>

$$p_i(t_1^-) = p_i(t_1^+) - \mu(t_1^+) \frac{\partial C}{\partial x_i} (t_1, x_i) \quad \text{for} \quad i = 1, \ldots, n \ , \tag{10}$$

and

$$H(t_1^-) = H(t_1^+) + \mu(t_1^+) \frac{\partial C}{\partial t} (t_1, x_i) \ . \tag{11}$$

A similar result was first given by McIntyre and Paiewonsky [20]. However, there is a sign error in their equation (52). Furthermore, the dual variables can be taken to be continuous at an exit corner. Thus, we have that <u>at exit corner</u>

$$p_i(t_2^-) = p_i(t_2^+) \qquad \text{for} \quad i = 1, \ldots, n \ , \tag{12}$$

and

$$H(t_2^-) = H(t_2^+) \ . \tag{13}$$

It is readily shown that (12) and (13) imply that

$$\mu(t_2^-) = 0 \ . \tag{14}$$

b. Higher Order SVIC.

In the case of a k^{th} $(k>1)$ order SVIC, the first time derivative of the state-variable constraint which explicitly contains the control variable is the k^{th}. Thus, on a constrained subarc on which $C(t,x_i(t)) = 0$ for $0 \le t_1 \le t \le t_2 \le T$ the control is determined by

$$\frac{d^k C}{dt^k} = 0 \qquad \text{for} \quad t_1 < t < t_2 \,, \tag{15}$$

and we further have

$$\frac{d^n C}{dt^n} = 0 \qquad \begin{array}{l} \text{for} \quad t_1 < t < t_2 \,, \\ \\ \text{for} \quad n = 1,\ldots,k-1 \,. \end{array} \tag{16}$$

The Hamiltonian is then given by [8], [9]

$$H(t,x_i,p_i,u) = L(t,x_i,u) + \sum_{i=1}^{n} p_i f_i(t,x_i,u) - \mu(t) \frac{d^k C}{dt^k} \,, \tag{17}$$

where

$$\mu(t) \begin{cases} = 0 & \text{for} \quad C(t,x_i) < 0 \,, \\ \\ \ge 0 & \text{for} \quad C(t,x_i) = 0 \,, \end{cases}$$

and the adjoint equations for the dual variables are given by

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{\partial L}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} p_j + \mu(t) \frac{\partial}{\partial x_i} \left(\frac{d^k C}{dt^k} \right), \tag{18}$$

for $i = 1,\ldots,n$. Again, we assume that initial conditions are given for all the state variables (i.e. $x_i(t=0) = x_i^o$ for $i = 1,\ldots,n$). Then the boundary conditions at $t = T$ for the dual variables are again given by (5).

On a constrained subarc, the control is determined by (15). As derived by Bryson, Denham, and Dreyfus [9], a necessary condition of optimality on a constrained subarc is

$$\mu(t) \geq 0 \qquad \text{for } t_1 \leq t \leq t_2 \,, \tag{19}$$

where the Lagrange multiplier (t) is determined by

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial L}{\partial u} + \sum_{i=1}^{n} p_i \frac{\partial f_i}{\partial u} - \mu(t)\frac{\partial}{\partial u}\left(\frac{d^k C}{dt^k}\right). \tag{20}$$

It remains to discuss the corner conditions (see [9], [20]). <u>At an entrance corner</u> we have that

$$p_i(t_1^-) = p_i(t_1^+) + \nu_0 \frac{\partial C}{\partial x_i}(t_1) + \sum_{n=1}^{k-2} \nu_n \frac{\partial}{\partial x_i}\left(\frac{d^n C}{dt^n}\right) - u(t_1^+)\frac{\partial}{\partial x_i}\left(\frac{d^{k-1}C}{dt^{k-1}}\right) \qquad \text{for } i - 1,\ldots,n, \tag{21}$$

and

$$H(t_1^-) = H(t_1^+) - \nu_0 \frac{\partial C}{\partial t}(t_1) - \sum_{n=1}^{k-2} \nu_n \frac{\partial}{\partial t}\left(\frac{d^n C}{dt^n}\right) + u(t_1^+)\frac{\partial}{\partial t}\left(\frac{d^{k-1}C}{dt^{k-1}}\right), \tag{22}$$

where $_j$ for $j = 0,1,\ldots,k-2$ are undetermined constants (multipliers). (Again, our results (21) and (22) differ in sign

with the corresponding results given by McIntyre and Paiewonsky [20].) Finally, <u>at an exit corner</u> we simply have

$$p_i(t_2^-) = p_i(t_2^+) \qquad \text{for} \quad i = 1,\ldots,n \ , \tag{23}$$

and

$$H(t_2^-) = H(t_2^+) \quad , \tag{24}$$

which imply that

$$\mu(t_2^-) = 0 \ . \tag{25}$$

4. <u>Application of Theory to Tactical Allocation Problems</u>.

In this section we apply the theory discussed above to two tactical allocation problems. Both these problems have first order state variable inequality constraints. The first problem (tactical air-war campaign) is a one-sided version of R. Isaacs's "War of Attrition and Attack [16]." However, we shall consider versions for which previous analysis [16] does not lead to an optimal policy. The second problem is the fire programming problem studied by Isbell and Marlow [17]. (See also [25], in this report as Appendix A, where results are obtained in a heuristic fashion (the determination of boundary conditions for the dual variables following [17]) without the appropriate theory of SVIC.) We shall consider several versions of both these problems. We will not develop complete solutions (including synthesis of optimal policies as a function of

initial conditions) to these problems but concentrate on developing the basic necessary conditions of optimality.

a. Tactical Air-War Campaign.

We consider a one-sided version of R. Isaacs's "War of Attrition and Attack [16]." We consider the case when one of the combatant forces always flys ground support missions.

(1) Original Version of Isaacs.

We consider the problem facing the X-force commander

$$\underset{\phi(t)}{\text{maximize}} \int_0^T \{x(1-\phi) - y\}\,dt \quad \text{with } T \text{ specified,}$$

$$\text{subject to: } \frac{dx}{dt} = m_1 \,,$$

$$\frac{dy}{dt} = m_2 - \phi c_2 x \,,$$

$$x,y \geq 0 \quad \text{and} \quad 0 \geq \phi \geq 1 \,, \tag{26}$$

where

x and y are the number of X and Y aircraft, respectively,

c_2 is the rate at which one X aircraft shoots down Y aircraft (Lanchester attrition-rate coefficient),

m_1 and m_2 are replacement rates,

ϕ is the fraction of total X aircraft which fly counter-air missions (and hence shoot down aircraft).

Clearly, the constraint $x \geq 0$ can never be binding. The Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = x(1-\phi) - y + p_1 m_1 + p_2(m_2-\phi c_2 x)$$
$$+ \mu(t)(m_2-\phi c_2 x) \qquad (27)$$

where

$$\mu(t) \begin{cases} = 0 & \text{for } y > 0 , \\ \geq 0 & \text{for } y = 0 , \end{cases}$$

and p_1 is the dual variable corresponding to x. The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = -(1-\phi) + \phi c_2(p_2+\mu) , \qquad (28)$$

$$\frac{dp_2}{dt} = 1 , \qquad (29)$$

with boundary conditions at $t = T$

$$p_1(t=T) = 0 , \qquad (30)$$

since $x(T) > 0$ (assuming $T > 0$) and

$$p_2(t=T) = \nu_2 , \qquad (31)$$

where

$$\nu_2 \begin{cases} = 0 & \text{for } y(T) > 0 , \\ \geq 0 & \text{for } y(T) = 0 . \end{cases}$$

When $y > 0$, the control law is determined by the maximum principle. This leads to

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad (-p_2(t)) > \dfrac{1}{c_2} \\ 0 & \text{for} \quad (-p_2(t)) < \dfrac{1}{c_2} . \end{cases} \tag{32}$$

The possibility of a singular solution (see Appendix D for a further discussion is excluded by the fact that since $\frac{dp_2}{dt} \neq 0$, it is impossible to have $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ when $\frac{\partial H}{\partial \phi} = 0$. Hence, the extremal control (it is also optimal, since we shall see that there is only one extremal) is well-defined by the control law (32) almost everywhere in time (i.e. the extremal control is determined by (32) except at a finite number of isolated instants in time when $(-p_2(t)) = \frac{1}{c_2}$).

Furthermore, the locus of points for which $(-p_2(t)) = \frac{1}{c_2}$ (and $\phi^*(t)$ consequently changes from 1 to 0 as an extremal "penetrates" this surface) may be referred to as a "switching" or "transition" surface. The discontinuity in the control at such a locus implies that the trajectory in the state space has a corner (without any path constraint), i.e. a discontinuous change in the slope of the state variable trajectory. There is no tangent to the path at the switching time, although both the left-hand and right-hand derivatives do exist. It is well-known (see, for example, p. 125 of [8] and also references to the classical calculus of

variations in Section 3. above) that the following conditions must hold at such a corner

$$p_i(t^-) = p_i(t^+) \quad \text{for} \quad i = 1,\ldots,n \ ,$$

$$H(t^-) = H(t^+) \ ,$$

where t^- is the time just before the corner (left-hand limit). It should be noted that these vital corner conditions are never mentioned in [1], [2], [16], or [24]. However, in all these references they are always implicitly assumed. As we will presently see below, the corner conditions may take a different form at the entrance to a constrained subarc (i.e. problem with SVIC when Gamkrelidze's approach (see pp. 265-267 of [22]) is used), and this implicit assumption is then no longer valid.

On a constrained subarc on which $y(t) = 0$ for $t_1 \leq t \leq t_2$ the control is determined by $\frac{dy}{dt} = 0$ and hence

$$\phi^*(t) = \frac{m_2}{c_2 x(t)} \qquad \text{for} \quad t_1 < t < t_2 \ . \tag{33}$$

The multiplier $\mu(t)$ is determined by the condition $\frac{\partial H}{\partial \phi} = 0$ and hence

$$\mu(t) = -\frac{1}{c_2} - p_2(t) \ . \tag{34}$$

The condition that $\mu(t) \geq 0$ yields that on a constrained subarc we must have

$$(-p_2(t)) \geq \frac{1}{c_2} . \tag{35}$$

Differentiating (34) and combining with (29), we find that

$$\dot{\mu}(t) = -\frac{dp_2}{dt} = -1 < 0 , \tag{36}$$

so that Gamkrelidze's condition is always satisfied on a constrained subarc.

Denoting the time of an entrance corner by t_1 and that of an exit corner by t_2, the corner conditions (10), (11), (12), and (13) yield that

$$p_2(t_1^-) = -\frac{1}{c_2} , \tag{37}$$

$$p_2(t_1^+) = -\frac{1}{c_2} - \mu(t_1^+) , \tag{38}$$

and

$$p_2(t_2^-) = p_2(t_2^+) = -\frac{1}{c_2} \tag{39}$$

When there is an exit from a constrained subarc at $t = t_2$, then (using (14) and (36)) it is readily shown that (38) becomes

$$p_2(t_1^+) = -\frac{1}{c_2} - (t_2 - t_1) . \tag{40}$$

Having developed the basic necessary conditions of optimality, let us now consider the synthesis of the optimal control. By the

synthesis of optimal control, we mean the explicit determination of the time history of the optimal control from initial to terminal time as a function of the initial state of the system. In synthesizing an optimal trajectory there are two cases to be considered:

$$\text{Case (a)} \qquad y(T) = 0 \ ,$$

$$\text{Case (b)} \qquad y(T) > 0 \ .$$

We now show that an optimal trajectory cannot end at $t = T$ with $y(T) = 0$.

For Case (a): $y(T) = 0$, there are two further subcases, corresponding to whether or not the trajectory was on a constrained subarc for a finite period of time immediately before the end at $t = T$. Thus, we consider

<u>Subcase (i) $y(T) = 0$ with $y(t) = 0$ for $t_1 \leq t \leq T$ where $t_1 < T$.</u>

In this case

$$p_2(t=T) = -\frac{1}{c_2} - \mu(t=T) \ , \tag{41}$$

where

$$\mu(t=T) \geq 0 \ ,$$

since we are on a constrained subarc. However, it is readily shown that this is incompatible with (31) which may be re-stated as

$$p_2(t=T) = \nu_2 \tag{42}$$

where

$$\nu_2 \geq 0 \ .$$

Hence, Subcase (i) is impossible by the inconsistency of (41) and (42), both of which are necessary conditions of optimality.

<u>Subcase (ii) $y(T) = 0$ with $y(t) > 0$ for $T - \delta \leq t < T$ where $\delta > 0$.</u>

Since $y(T) = 0$ with $y(t) > 0$ for $T - \delta \leq t < T$ where $\delta > 0$, we must have (considering the state equations (26))

$$\phi^*(t) = 1 \qquad \text{for} \quad T - \delta \leq t < T \ . \tag{43}$$

By (31) (or (42)) and (29), we have

$$p_2(t) = \nu_2 - (T-t) \tag{44}$$

where

$$\nu_2 \geq 0 \ .$$

Thus, we can find $\varepsilon > 0$ such that for $T - \varepsilon \leq t < T$, we have

$$p_2(t) > \frac{1}{c_2} \ , \tag{45}$$

and hence considering the control law (32), we are led to a contradiction of (43). Thus, Subcase (ii) is similarly impossible. Hence, optimal trajectories are only possible in Case (b). It had not been shown previously [16] that Case (a) is not consistent with an optimal

policy. (In fact, the possibility of Case (a) is not even considered in [16], [24].)

For Case (b): $y(T) > 0$, we have by (31) that

$$p_2(t=T) = 0 \ . \tag{46}$$

Using this terminal condition, a backwards integration of the adjoint equation (29) yields

$$(-p_2(\tau)) = \tau \ , \tag{47}$$

where, for convenience, we have introduced the backwards time τ defined by $\tau = T-t$. Hence, in the case when $y(t) > 0$ for all t, it is easily seen that

$$\phi^*(\tau) = \begin{cases} 1 & \text{for } \tau > \dfrac{1}{c_2} \ , \\ 0 & \text{for } \tau < \dfrac{1}{c_2} \ , \end{cases} \tag{48}$$

where we have considered (32) and (47). Thus, in this special case (when $y(t) > 0$ for $0 \le t \le T - \frac{1}{c_2}$) the solution shown in Table I readily follows. Combining the solution shown in Table I with a forward integration of the state equations (26), it is easy to show that this solution applies under the following conditions:

$$\text{either (a)} \quad m_1 c_2 (T - \frac{1}{c_2}) < m_2 - c_2 x_0 \ , \tag{49}$$

$$\text{or (b)} \begin{cases} m_1 c_2 (T - \dfrac{1}{c_2}) \ge m_2 - c_2 x_0 \ , \\ y_0 > \dfrac{c_2 m_1}{2} (T - \dfrac{1}{c_2})^2 + (c_2 x_0 - m_2)(T - \dfrac{1}{c_2}) \ . \end{cases} \tag{50}$$

Table I. Optimal Policy in Tactical Air-War Campaign When $y(t) > 0$ for all t.

Case (1): $T \leq \frac{1}{c_2}$

$$\phi^*(t) = 0 \qquad \text{for} \quad 0 \leq t \leq T .$$

(Always fly ground support.)

Case (2): $T > \frac{1}{c_2}$

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq T - \frac{1}{c_2} , \\ 0 & \text{for} \quad T - \frac{1}{c_2} \leq t \leq T . \end{cases}$$

It remains to develop the solution in Case (b) when neither (49) nor (50) holds. Thus when

$$y_0 \leq \frac{c_2 m_1}{2}\left(T - \frac{1}{c_2}\right) + (c_2 x_0 - m_2)\left(T - \frac{1}{c_2}\right), \tag{51}$$

we have $y(t_1) = 0$ where $t_1 < T - \frac{1}{c_2}$. In this case $y(t) > 0$ for $0 \leq t < t_1$ and recalling (37) that $p_2(t_1^-) = -\frac{1}{c_2}$, it is readily seen that $\phi^*(t) = 1$ for $0 \leq t < t_1$. Furthermore,

$$\phi^*(t) = \frac{m_2}{c_2\, x} \qquad \text{for} \quad t_1 \leq t \leq T - \frac{1}{c_2}. \tag{52}$$

We observe that by (40)

$$p_2(t_1^+) = -\frac{1}{c_2} - \left(T - \frac{1}{c_2} - t_1\right). \tag{53}$$

Clearly (35) is satisfied. Thus, both (8) and (9) are satisfied so that it is optimal "in the small" to remain on the constrained subarc. Finally, it is readily shown that (39) and (46) are consistent. The final solution (the extremal trajectory is optimal, since it is unique) may be expressed in a particularly simple form and is shown in Table II.

(2) <u>A Time-Dependent Attrition-Rate Coefficient</u>.

If the approach used by Isaacs [16] is applied to the above problem, then it is assumed that it is optimal to drive $y(t)$ to zero and keep it there for $t \leq T - \frac{1}{c_2}$ whenever possible. However,

Table II. Optimal Policy in

Tactical Air-War Campaign

Case (1): $T \le \frac{1}{c_2}$

$$\phi^*(t) = 0 \qquad \text{for} \quad 0 \le t \le T \ .$$

(Always fly ground support.)

Case (2): $T > \frac{1}{c_2}$

For $0 \le t \le T - \frac{1}{c_2}$

$$\phi^*(t) = \begin{cases} 1 & \text{when} \quad y > 0 \ , \\ \frac{m_2}{c_2 x} & \text{when} \quad y = 0 \ . \end{cases}$$

For $T - \frac{1}{c_2} \le t \le T$, $\quad \phi^*(t) = 0$.

this need not be true and, indeed, is not true in general as the following example shows. We consider the case when the attrition-rate coefficient c_2 is a function of time.

$$\underset{\phi(t)}{\text{maximize}} \int_0^T \{x(1-\phi) - y\}\, dt \qquad \text{with } T \text{ specified},$$

$$\text{subject to: } \frac{dx}{dt} = m_1 ,$$

$$\frac{dy}{dt} = m_2 - \phi\, c_2(t)\, x ,$$

$$x,y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1 . \tag{54}$$

We will consider only the development of the basic necessary conditions of optimality. The analysis of the preceeding section applied through (35). However, c_2 is now a function of time. Re-writing (34) as

$$\mu(t) = -\frac{1}{c_2(t)} - p_2(t) , \tag{55}$$

we have

$$\dot{\mu}(t) = \frac{1}{(c_2)^2}\frac{dc_2}{dt} - 1 , \tag{56}$$

so that Gamkrelidze's condition (9) is only satisfied on a constrained subarc when

$$\frac{dc_2}{dt} \leq [c_2(t)]^2 . \tag{57}$$

The corner conditions (10) through (13) now take the form

$$p_2(t_1^-) = -\frac{1}{c_2(t_1)} , \tag{58}$$

$$p_2(t_1^+) = -\frac{1}{c_2(t_1)} - \mu(t_1^+) , \tag{59}$$

and

$$p_2(t_2^-) = p_2(t_2^+) = -\frac{1}{c_2(t_2)} . \tag{60}$$

Considering (14) and (56), it is easily seen that

$$\mu(t) = -\int_t^{t_2} \left\{\frac{1}{(c_2)^2}\frac{dc_2}{dt} - 1\right\} dt ,$$

and hence for $t_1 \leq t \leq t_2$

$$\mu(t) = t_2 - t + \frac{1}{c_2(t_2)} - \frac{1}{c_2(t)} . \tag{61}$$

Considering (61), we see that when there is an exit from a constrained subarc at $t = t_2$, we have

$$p_2(t_1^+) = -\frac{1}{c_2(t_2)} - (t_2-t_1) . \tag{62}$$

(3) Linearly Increasing Attrition-Rate Coefficient.

It seems appropriate to consider a special case of a time-dependent attrition-rate coefficient in order to more clearly illustrate that there are restrictions on when it is optimal to keep $y(t)$ equal to zero for a finite interval of time. Let us therefore consider the special case when $c_2(t)$ is a linear function of time, i.e.

$$c_2(t) = kt \ . \tag{63}$$

In this case, (57) holds (this implies that Gamkrelidze's necessary condition (9) is satisfied on a constrained subarc) only for

$$t \geq \frac{1}{\sqrt{k}} \ . \tag{64}$$

Furthermore, (58) now becomes

$$p_2(t_1^-) = -\frac{1}{kt_1} \ , \tag{65}$$

so that by using (29) we find that for $0 \leq t \leq t_1$

$$p_2(t) = -\frac{1}{kt_1} + t - t_1 \ . \tag{66}$$

Considering the control law (32), the above expression (66) yields that

$$\phi^*(t) = \begin{cases} 0 & \text{for } 0 \le t < \dfrac{1}{kt_1} , \\ 1 & \text{for } \dfrac{1}{kt_1} < t < t_1 . \end{cases} \tag{67}$$

Let us denote the time of switch in tactics by t_s so that we have

$$t_s = \frac{1}{kt_1} . \tag{68}$$

Integration of the state equations (54) using (67) then yields

for $t_s \le t \le t_1$

$$y(t) = y_0 + m_2 t - \frac{x_0 k}{2}(t^2 - t_s^2) - \frac{m_1 k}{3}(t^3 - t_s^3) . \tag{69}$$

The unknown quantity t_1 can then be determined by solving the equation

$$y(t_1) = 0 , \tag{70}$$

where $y(t)$ is given by (69) and t_s is given by (68). We have not worked out further details.

Furthermore, if we are never on a constrained subarc then

$$\phi^*(t) = \begin{cases} 1 & \text{for } T-t > \dfrac{1}{kt} \\ 0 & \text{for } T-t < \dfrac{1}{kt} . \end{cases} \tag{71}$$

We observe that

$$\phi^*(t=0) = 0 \ ,$$

and

$$\phi^*(t=T) = 0 \ .$$

If there is a switch in tactics, then (ignoring the pathological case) there must be two, which we denote as t_{s_1} and t_{s_2}. These are the roots of the equation

$$T - t_s = \frac{1}{kt_s} \ . \tag{72}$$

Consequently, for $T \le \frac{2}{\sqrt{k}}$ we have

$$\phi^*(t) = 0 \qquad \text{for} \quad 0 \le t \le T \ . \tag{73}$$

Also, for $T > \frac{2}{\sqrt{k}}$ we have

$$\phi^*(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < t_{s_1} \ , \\ 1 & \text{for} \quad t_{s_1} < t < t_{s_2} \ , \\ 0 & \text{for} \quad t_{s_2} < t \le T \ , \end{cases} \tag{74}$$

where

$$t_{s_1} = \frac{T}{2} - \sqrt{\left(\frac{T}{2}\right)^2 - \frac{1}{k}} \ , \tag{75}$$

and

$$t_{s_2} = \frac{T}{2} + \sqrt{\left(\frac{T}{2}\right)^2 - \frac{1}{k}} \ . \tag{76}$$

It seems appropriate to briefly discuss what this partial analysis has yielded. First of all, we (the X-forces) don't begin the campaign by shooting down Y aircraft but wait until our effectiveness rises enough over time. Intuitively we may think of this as manifesting that we should not shoot down planes today (initially) because we can do so much better tomorrow. Secondly, we adjust our tactics so that (64) holds when we keep the number of Y aircraft at the zero level. We note that we don't always drive $y(t)$ to zero whenever we can, but we wait so that (64) can hold. We have not worked out the conditions on the initial force levels under which a constrained subarc occurs.

(4) Time Variations in Integrand of Payoff.

Let us further consider an example in which the payoff from allocating aircraft to ground support missions can change over time. In other words, we try to take account of changing effectiveness of the supporting weapon system (here aircraft). The model is

$$\underset{\phi(t)}{\text{maximize}} \int_0^T \{a(t)x(1-\phi) - b(t)y\}\, dt \quad \text{with } T \text{ specified} ,$$

$$\text{subject to: } \frac{dx}{dt} = m_1 ,$$

$$\frac{dy}{dt} = m_2 - \phi c_2 x ,$$

$$x,y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1 , \tag{77}$$

where c_2 is again assumed to be constant.

Again, we shall focus on the development of basic necessary conditions of optimality and an interpretation of the parameters on which the optimal policy depends. The Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = a(t)x(1-\phi) - b(t)y + p_1m_1 + p_2(m_2-\phi c_2x)$$

$$+ \mu(t)(m_2-\phi c_2x) \quad , \tag{78}$$

where

$$\mu(t) \begin{cases} = 0 & \text{for} \quad y > 0 \ , \\ \geq 0 & \text{for} \quad y = 0 \ , \end{cases}$$

and the adjoint equations for the dual variables are

$$\frac{dp_1}{dt} = - a(t)(1-\phi) + \phi c_2(p_2+\mu) \quad , \tag{79}$$

$$\frac{dp_2}{dt} = b(t) \quad , \tag{80}$$

with boundary conditions at $t = T$ again given by (30) and (31).

When $y > 0$, the control law is determined by the maximum principle. This leads to

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad (-p_2(t)) > \dfrac{a(t)}{c_2} \ , \\ 0 & \text{for} \quad (-p_2(t)) < \dfrac{a(t)}{c_2} \ . \end{cases} \tag{81}$$

The probability of a singular solution, however, is not as easily excluded as before. Although it is readily shown to be possible to have $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ when $\frac{\partial H}{\partial \phi} = 0$, complete details as to whether there can be a singular subarc haven't been worked out at this time.

Again, on a constrained subarc on which $y(t) = 0$ for $t_1 \le t \le t_2$ the control to keep $y(t) = 0$ is given by (33). The multiplier $\mu(t)$ is determined by the condition $\frac{\partial H}{\partial \phi} = 0$ and hence

$$\mu(t) = -\frac{a(t)}{c_2} - p_2(t) \ . \tag{82}$$

The condition that $\mu(t) \ge 0$ yields that on a constrained subarc we must have

$$(-p_2(t)) \ge \frac{a(t)}{c_2} \tag{83}$$

Differentiating (82) and combining with (80), we find that

$$\dot{\mu}(t) = -\frac{1}{c_2}\frac{da}{dt} - b(t) \ , \tag{84}$$

so that Gamkrelidze's condition (9) is only satisfied on a constrained subarc when

$$b(t) \ge -\frac{1}{c_2}\frac{da}{dt} \ . \tag{85}$$

The corner conditions (10) through (13) now yield

$$p_2(t_1^-) = -\frac{a(t_1)}{c_2}\,, \qquad . \tag{86}$$

$$p_2(t_1^+) = -\frac{a(t_1)}{c_2} - \mu(t_1^+)\,, \tag{87}$$

and

$$p_2(t_2^-) = p_2(t_2^+) = -\frac{a(t_2)}{c_2}\,. \tag{88}$$

Considering (14) and (84), it is easily seen that for $t_1 \le t \le t_2$

$$\mu(t) = \frac{1}{c_2}\{a(t_2) - a(t)\} + \int_t^{t_2} b(s)\,ds\,, \tag{89}$$

and thus when there is an exit from a constrained subarc at $t = t_2$, we have

$$p_2(t_1^+) = -\frac{a(t_2)}{c_2} - \int_{t_1}^{t_2} b(s)\,ds\,. \tag{90}$$

From the above analysis we see that the "weight" of the ground support missions, i.e. $a(t)$ and $b(t)$, are determining factors in ascertaining optimal allocation tactics. From this we conclude that another type of model (one that considers the consequence of allocating aircraft to ground support missions) may be more appropriate. In particular, the model (26) (due to A. Mengel)

ignores the dynamics of ground combat (i.e. the interaction of the supporting X and Y weapon systems with the battle of ground forces). We propose this to ONR as a future research task.

b. The Isbell-Marlow Fire Programming Problem.

We consider a problem about the optimal distribution of fire over enemy target types. This problem was first studied by Isbell and Marlow [17] (the same analysis is apparently given in [24]). However, we have shown that the solution given in [17] is not valid for all ranges of model parameters [25] (reproduced in this report as Appendix A). (There are some gaps in our solution development that we rectify in Appendix F, although our previous solution [25] is not modified.) Additionally, we developed in [25] a purely algebraic method for constructing a solution to such problems, as opposed to the geometric method (which failed to uncover a completely correct solution) employed in [17].

However, in [25] we still used the same heuristic method introduced by Isbell and Marlow [17] to determine boundary conditions for the dual variables. There are some other technical gaps having to do with the treatment of the non-negativity of the force levels (state variable inequality constraints). We re-consider the development of certain key necessary conditions of optimality within the framework of the theory of SVIC. This is the first such application of the theory of SVIC to such problems to the best of our knowledge.

(1) Original Version.

We consider the problem

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ unspecified},$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2,$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where

$x_1(t)$, $x_2(t)$, $y(t)$ are force levels,

p, q, r are utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

and ϕ is the fraction of Y fire directed at X_1.

The battle terminates at $t = T$ with either (a) $x_1(T) = x_2(T) = 0$ or (b) $y(T) = 0$ (i.e. it is a "fight to the finish"). As in [25] we shall denote the five terminal states of battle corresponding to the above two conditions as C_i for $i = 1,\ldots,5$ (see [25] for precise definitions).

We will focus on the development of necessary conditions of optimality. The synthesis of optimal trajectories from these is

carried out in Appendix A (see also Appendix F). The Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = -p_1\phi a_1 y - p_2(1-\phi)a_2 y - p_3(b_1x_1+b_2x_2)$$

$$- \mu_1(t)\phi a_1 y - \mu_2(t)(1-\phi)a_2 y \ , \tag{92}$$

where

$$\mu_i(t) \begin{cases} = 0 & \text{for } x_i > 0 \ , \\ \geq 0 & \text{for } x_i = 0 \ . \end{cases}$$

The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = b_1 p_3 \ , \tag{93}$$

$$\frac{dp_2}{dt} = b_2 p_3 \ , \tag{94}$$

$$\frac{dp_3}{dt} = \phi a_1(p_1+\mu_1) + (1-\phi)a_2(p_2+\mu_2) \ . \tag{95}$$

Deferring the discussion of boundary conditions for the dual variables until later, we note the transversality condition

$$H(t=T,x_i,p_i^*,\phi^*) = 0 \ . \tag{96}$$

When $x_1,x_2 > 0$, the control law is determined by the maximum principle. This leads to

$$\phi^*(t) = \begin{cases} 1 & \text{for } a_1(-p_1(t)) > a_2(-p_2(t)) , \\ 0 & \text{for } a_1(-p_1(t)) < a_2(-p_2(t)) . \end{cases} \tag{97}$$

As we showed in [25], it is easily seen that it is impossible to have $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ when $\frac{\partial H}{\partial \phi} = 0$ so that it is impossible to have a singular solution.

Without loss of generality, we may consider a constrained subarc on which $x_1(t) = 0$ (and $x_2 > 0$) for $t_1 \le t \le t_2$. The control is clearly $\phi^*(t) = 0$. The multiplier $\mu_1(t)$ is determined by the condition $\frac{\partial H}{\partial \phi} = 0$ and hence

$$\mu_1(t) = \frac{1}{a_1} (a_2p_2 - a_1p_1) . \tag{98}$$

The condition that $\mu_1(t) \ge 0$ yeilds that on a constrained subarc we must have

$$a_1(-p_1(t)) \ge a_2(-p_2(t)) . \tag{99}$$

Differentiating (98) and combining with (93) and (94), we find that

$$\dot{\mu}(t) = \frac{p_3(t)}{a_1} (a_2b_2 - a_1b_1) , \tag{100}$$

so that Gamkrelidze's condition (9) is only satisfied on a constrained subarc with $x_1 = 0$ when

$$a_1b_1 \ge a_2b_2 , \tag{101}$$

since it is readily shown that $p_3(t) \geq 0$. The corner conditions (10) and (11) at an entrance corner yield that

$$p_1(t_1^-) = p_1(t_1^+) + \mu_1(t_1^+) \ , \tag{102}$$

$$p_2(t_1^-) = p_2(t_1^+) \ , \tag{103}$$

$$p_3(t_1^-) = p_3(t_1^+) \ , \tag{104}$$

and

$$p_1(t_1^-) = \frac{a_2}{a_1} p_2(t_1^-) \ . \tag{105}$$

Let us now consider the boundary conditions on the dual variables at $t = T$. We use the same notation for the terminal states of battle as used in [25] (reproduced in this report as Appendix A). <u>Without loss of generality, we can assume that $a_1b_1 > a_2b_2$</u>. Then Gamkrelidze's condition (101) implies that no extremals can lead to C_3, and it may be shown that there is no inconsistency with having extremals lead to C_2. Terminal states C_1, C_4, and C_5 are treated in a common fashion, since for all of them the time of the end of battles is determined by $y(T) = 0$. Hence, $y(T) = 0$ is a specified equality condition on a state variable at an unspecified terminal time. Considering this fact (see p. 75 of [8]) and (5), we have for states C_1, C_4, and C_5

$$p_1(t=T) = -p + \nu_1 \tag{106}$$

where

$$\nu_1 \begin{cases} = 0 & \text{for } x_1 > 0 \, , \\ \geq 0 & \text{for } x_1 = 0 \, , \end{cases}$$

$$p_2(t=T) = -q + \nu_2 \tag{107}$$

where

$$\nu_2 \begin{cases} = 0 & \text{for } x_2 > 0 \, , \\ \geq 0 & \text{for } x_2 = 0 \, , \end{cases}$$

$$p_3(t=T) = \nu_3 \tag{108}$$

where ν_3 is unrestricted.

Furthermore, the transversality condition (96) yields in all these cases (i.e. for C_1, C_4, and C_5)

$$p_3(t=T) = 0 \, . \tag{109}$$

In particular for C_4: $x_1(T) > 0$, $x_2(T) = 0$, $y(T) = 0$, it is readily seen by (101) that this terminal state cannot be reached by being on a constrained subarc with $x_2 = 0$ for a finite interval of time. Thus, C_4 can only be reached when $x_2(T) = 0$ but $x_2(t) > 0$ for $t < T$. The boundary conditions (106) and (107) on the dual variables now take the form

$$p_1(t=T) = -p \ , \tag{110}$$

and

$$p_2(t=T) = -q + \nu_2 \ , \tag{111}$$

where $\nu_2 \geq 0$.

The latter condition (111) was conjectured to hold in [25], but we could not at that time justify it. Since $x_2(T) = 0$ but $x_2(t) > 0$ for $t < T$, we must have $\phi^*(T) = 0$, and hence by (97) we have

$$(-p(t=T)) > \frac{a_1 p}{a_2} \ . \tag{112}$$

Then, combination of (111) and (112) yields that

$$\frac{a_1 p}{a_2} < (-p_2(t=T)) \leq q \ , \tag{113}$$

which was a result used in [25] that we could not justify (due to our, at that time, lack of knowledge of the theory of SVIC).

(2) <u>A Case of Time-Dependent Attrition-Rate Coefficients</u>.

As we discussed in [27], for problems with autonomous dynamics (8) may well imply that Gamkrelidze's condition (9) is satisfied. In non-autonomous cases this need not be the case and highlights the importance of Gamkrelidze's condition. This work here complements our results presented in [26] (reproduced in this report as Appendix C),

where we omitted considerations pertaining to the non-negativity of the force levels. Let us therefore consider the problem

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ specified ,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1(t)y + r_1(t) \ ,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2(t)y + r_2(t) \ ,$$

$$\frac{dy}{dt} = -b_1(t)x_1 - b_1(t)x_2 + s(t) \ ,$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1 \ , \tag{114}$$

where $r_1(t)$, $r_2(t)$, $s(t)$ are replacement rates. Again, we focus on the development of some necessary conditions of optimality.

The Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = p_1(t)\{-\phi a_1(t) + r_1(t)\} +$$

$$p_2(t)\{-(1-\phi)a_2(t)y + r_2(t)\} + p_3(t)\{-b_1(t)x_1 - b_2(t)x_2 + s(t)\}$$

$$+ \mu_1(t)\{-\phi a_1(t)y + r_1(t)\} + \mu_2(t)\{-(1-\phi)a_2(t)y + r_2(t)\} \ , \tag{115}$$

where

$$\mu_i(t) \begin{cases} = 0 & \text{for } x_i > 0 \ , \\ \geq 0 & \text{for } x_i = 0 \ . \end{cases}$$

Equations (93) through (97) still hold with a_1, a_2, b_1, and b_2 now being functions of time.

On a constrained subarc on which $x_1(t) = 0$ (and $x_2 > 0$) for $t_1 \le t \le t_2$ the control is determined by $\frac{dx_1}{dt} = 0$ so that

$$\phi^*(t) = \frac{r_1(t)}{a_1(t)y} \qquad \text{for} \quad t_1 \le t \le t_2 \ . \tag{116}$$

The multiplier $\mu_1(t)$ is determined by the condition $\frac{\partial H}{\partial \phi} = 0$ and hence

$$\mu_1(t) = \frac{1}{a_1(t)} \{a_2(t)p_2(t) - a_1(t)p_1(t)\} \ . \tag{117}$$

The condition that $\mu_1(t) \ge 0$ yields that on a constrained subarc we must have

$$a_1(t)(-p_1(t)) \ge a_2(t)(-p_2(t)) \ . \tag{118}$$

Similar to what we did in [26], let us now consider the special case when $a_i(t) = k_{a_i} g(t)$ for $i = 1,2$. Then (117) becomes

$$\mu_1(t) = \frac{1}{k_{a_1}} (k_{a_2} p_2 - k_{a_1} p_1) \ . \tag{119}$$

Differentiating (119) with respect to time and combining with (93) and (94), we find that

$$\dot{\mu}_1(t) = \frac{p_3(t)}{a_1(t)} \{a_2(t)b_2(t) - a_1(t)b_1(t)\} \ , \tag{120}$$

$$a_1(t)b_1(t) \geq a_2(t)b_2(t) \quad , \tag{121}$$

since again $p_3(t) \geq 0$. The corner conditions (102) through (105) still hold only with a_1 and a_2 being functions of time.

(3) Several Target Types.

In [26] (reproduced in this report as Appendix C) we first considered a scenario with one weapon system in battle against n target types. Our previous treatment ignored the non-negativity restrictions on the force levels. To rectify this gap we consider

$$\underset{\phi_i(t)}{\text{maximize}} \quad \{vy(T) - \sum_{i=1}^{n} w_i x_i(T)\} \quad \text{with } T \text{ specified} ,$$

$$\text{subject to:} \quad \frac{dx_i}{dt} = -\phi_i a_i y \quad \text{for } i = 1,\ldots,n ,$$

$$\frac{dy}{dt} = - \sum_{i=1}^{n} b_i x_i ,$$

$$x_1,\ldots,x_n,y \geq 0 , \quad \phi_i \geq 0 \quad \text{for } i = 1,\ldots,n ,$$

$$\text{and} \quad \sum_{i=1}^{n} \phi_i = 1 . \tag{122}$$

We focus on the development of necessary conditions of optimality. To avoid being encumbered by too many symbols, we will consider the case when the X_1 - force is driven to zero and assume that $x_2,\ldots, x_n > 0$. In this case, however, we must consider a

slightly different form for the Hamiltonian. (See pp. 108-109 in [8] for a discussion of the difference between Hamiltonians denoted there as H and H^*; the Hamiltonian given by (123) is H in the notation of [8]. As pointed out in [8], whether one uses a Hamiltonian of the form (92) or (123) is a matter of taste.)

$$H(t,x_i,p_i,\phi_i) = y\{\sum_{i=1}^{n} a_i(-p_i)\phi_i\} - p_{n+1}\sum_{i=1}^{n} b_i x_i + \lambda(1 - \sum_{i=1}^{n} \phi_i) + \mu(-\phi_1 a_1 y) + \sum_{i=1}^{n} \gamma_i\phi_i \,, \tag{123}$$

where

$$\mu(t)\begin{cases} = 0 & \text{for } x_1 > 0 \,, \\ \geq 0 & \text{for } x_1 = 0 \,, \end{cases}$$

and

$$\gamma_i\begin{cases} = 0 & \text{for } \phi_i > 0 \,, \\ \geq 0 & \text{for } \phi_i = 0 \,. \end{cases}$$

The adjoint system of differential equations for the dual variables is

$$\frac{dp_i}{dt} = b_i p_{n+1} \qquad \text{for } i = 1,\ldots,n \,, \tag{124}$$

and

$$\frac{dp_{n+1}}{dt} = \phi_1 a_1(p_1+\mu_1) + \sum_{i=2}^{n} \phi_i a_i p_i \,. \tag{125}$$

On a constrained subarc on which $x_1(t) = 0$ (and $x_2, \ldots, x_n > 0$) for $t_1 \le t \le t_2$, part of the control is clearly $\phi_1^*(t) = 0$. To determine the multiplier $\mu(t)$, we must take a slightly different approach than previously used above. This development will show the reader the origin of the form of the Hamiltonian (123).

Following Gamkrelidze (see pp. 265-267 of [22]), on a constrained subarc the maximum principle takes the following form

$$\underset{\phi_i}{\text{maximize}} \quad h(t, x_i, p_i, \phi_i)$$

$$\text{subject to:} \quad \sum_{i=1}^{n} \phi_i = 1 \; ,$$

$$\phi_i \ge 0 \qquad \text{for } \; i = 1, \ldots, n \; ,$$

$$\frac{dx_1}{dt} = -\phi_1 a_1 y \ge 0 \; , \tag{126}$$

where

$$h(t, x_i, p_i, \phi_i) = y\{\sum_{i=1}^{n} a_i(-p_i)\phi_i\} - p_{n+1} \sum_{i=1}^{n} b_i x_i \; . \tag{127}$$

We re-write (126) in a more convenient form to apply the well-known Kuhn-Tucker conditions [19]. (Since the constraints are linear, it is well-known that the Kuhn-Tucker constraint qualification [19] is satisfied. See [2] for a further discussion.)

$$\underset{\phi_i}{\text{maximize}} \quad h(t,x_i,p_i,\phi_i)$$

$$\text{subject to:} \quad \sum_{i=1}^{n} \phi_i = 1 \ ,$$

$$-\phi_i \le 0 \quad \text{for} \quad i = 1,\ldots,n \ ,$$

$$\phi_1 a_1 y \le 0 \ , \tag{128}$$

Then the Lagrangian for the optimization problem (128) on the constrained subarc (which turns out to be the Hamiltonian (123)) is given by

$$L(\phi_i,\mu,\gamma_i,\nu_i) = h(t,x_i,p_i,\phi_i) + \lambda(1-\sum_{i=1}^{n}\phi_i)$$

$$+ \mu(-\phi_1 a_1 y) + \sum_{i=1}^{n} \gamma_i \phi_i \tag{129}$$

$$= H(t,x_i,p_i,\phi_i) \ .$$

Applying the well-known Kuhn-Tucker conditions [19] to (128), we have that necessary and sufficient conditions for the maximum to the concave program (128) are

$$\frac{\partial L}{\partial \phi_1} = -p_1 a_1 y - \lambda - \mu a_1 y + \gamma_1 = 0 \ , \tag{130}$$

$$\frac{\partial L}{\partial \phi_i} = -p_i a_i y - \lambda + \gamma_i = 0 \quad \text{for} \quad i = 2,\ldots,n \ , \tag{131}$$

$$\mu(\phi_1^* a_1 y) = 0 \ , \tag{132}$$

$$\gamma_i \phi_i^* = 0 \quad \text{for} \quad i = 1,\ldots,n \ , \tag{133}$$

and

$$\mu, \gamma_i \geq 0. \tag{134}$$

Let us assume that there is an isolated maximum to (128) (thereby ignoring the troublesome case of alternate optima, which merely presents inessential complexities; it is readily shown that (128) cannot have alternate optima except at isolated points in time). Then we let $i = J$ be the index such that $\phi_J^* = 1$. By the complementary slackness relationship (133), we have $\gamma_J = 0$ so that (131) yields

$$\lambda = (-p_J) a_J \ y. \tag{135}$$

Using (135) we may re-write the n equations (130) and (131) as

$$\mu \ a_1 y - \gamma_1 = y(a_J p_J - a_1 p_1), \tag{136}$$

and

$$\gamma_i = y\{a_J(-p_J(t) - a_i(-p_i(t))\} \quad \text{for} \quad i = 2,\ldots,n. \tag{137}$$

Assuming that $y > 0$ (otherwise there is no allocation problem), the Kuhn-Tucker optimality condition $\gamma_i \geq 0$ for $i = 1,\ldots,n$ yields

$$a_J(-p_J(t)) \geq a_i(-p_i(t)) \quad \text{for} \quad i = 2,\ldots,n. \tag{138}$$

In fact, for the maximum to be isolated, we must have

$$a_J(-p_J(t)) > a_i(-p_i(t)) \quad \text{for} \quad i = 2,\ldots,n \ \text{and} \ i \neq J. \tag{139}$$

We are therefore left with one equation for the two remaining undertermined multipliers μ and γ_1

$$\mu\, a_1 y - \gamma_1 = y(a_J p_J - a_1 p_1). \tag{140}$$

Let us note that once we reach the constrained subarc and $x_1(t=t_1) = 0$, $x_1(t)$ remains equal to zero for $t_1 \le t \le t_2 = T$, since the model (122) does not allow for replacements. Thus by Gamkrelidze's device the constraint $\phi_1 a_1 y \le 0$ is active for $t_1 < t \le T$. Observing that the constraint $\phi_1 a_1 y \le 0$ being active guarantees that the constraint $\phi_1 \ge 0$ is satisfied (we note that $a_1 > 0$ and $y > 0$), we set $\gamma_1 = 0$ to obtain

$$\mu(t) = \frac{1}{a_1}(a_J p_J - a_1 p_1). \tag{141}$$

Then the Kuhn-Tucker optimality condition $\mu \ge 0$ implies that

$$a_1(-p_1(t)) \ge a_J(-p_J(t)), \tag{142}$$

on the constrained subarc. Differentiating (141) and combining the result with the adjoint equations (124), we obtain

$$\dot{\mu}(t) = \frac{p_{n+1}(t)}{a_1}(a_J b_J - a_1 b_1), \tag{143}$$

so that Gamkrelidze's condition (9) is only satisfied on a constrained subarc with $x_1 = 0$ when

$$a_1 b_1 \ge a_J b_J, \tag{144}$$

since it is readily shown that $p_{n+1}(t) > 0$ for $0 \le t < T$.

Thus, we have proved the following theorem:

> Theorem E.1: For the problem (122), an optimal policy has the property that if we annihilate X_1, then we must have
>
> $$a_1 b_1 \geq a_J b_J \ ,$$
>
> where J is the index of any target type upon which fire is later concentrated.

Although the above does not complete the analysis of this problem, we end our discussion here. We propose to ONR that the complete discussion of the conditions under which it is optimal to drive a force level to zero be a future research task.

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Appendix F. The Isbell-Marlow Problem Revisited.

1. Introduction.

We have included in this report as Appendix A some of our previous work [9] on the Isbell-Marlow fire programming problem [6] (the same results are apparently given in [7]). This work [9] revises some even earlier work (see Appendix A of [8]) that we had done on this problem. Since the writing of [9] in June 1971, we have uncovered a gap in our development of the synthesis of the optimal control (however, the solution presented in [9] remains valid). (By the synthesis of optimal control, we mean using the basic necessary conditions of optimality to explicitly determine the time history of the optimal control from initial to terminal time as a function of the initial state of the system.) Thus, it is the purpose of this appendix to revise our earlier work [9] (repreduced in this report as Appendix A). The results we obtain from this further examination of the Isbell-Marlow problem are of considerable interest, since they provide an understanding of solution phenomena (multiple extremals and a dominated payoff) that we have encountered in a differential game [10].

In [9] we showed that extremals were not unique for a certain range of model parameters and tried to develop necessary and sufficient conditions for optimal paths to lead to all the terminal states of combat. (By an extremal we mean a battle trajectory on which the necessary conditions of optimality are almost everywhere satisfied in time.) In the pioneering 1956 work of Isbell and Marlow [6] (the Pontryagin maximum principle was only announced in 1956 [1], although

Professor Magnus Hestenes of the USA had apparently first given the now standard control formulation (as well as developed optimality conditions) and urged others to follow his approach in 1949 [3]) one can find several gaps (boundary conditions for dual variables, treatment of state variable inequality constraints (SVIC)) in the light of subsequent control theory developments. In [9] we followed Isbell and Marlow's heuristic treatment of the determination of boundary conditions for the dual variables and did not consider the theory of SVIC. We rectify these gaps here.

Furthermore, we feel that one of our major contributions in [9] was the development of a purely algebraic method for the determination of the optimal control, as opposed to the geometric method employed in [6]. The reader should note the difficulty of applying their geometric approach to a similar problem with a state space of dimension greater than three. We feel that this is essentially impossible, whereas our algebraic method (like algebraic geometry in n-dimensional space) appears to be readily applicable to such problems.

2. Statement of the Isbell-Marlow Fire Programming Problem.

For the reader's convenience we re-state the problem originally studied by Isbell and Marlow [6]

$$\underset{\phi(t)}{\text{maximize}} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ unspecified,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 y,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2, \tag{1}$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where all symbols are defined in the next section.

The battle terminates upon reaching the terminal states defined by (1) $x_1(T) = x_2(T) = 0$ and (2) $y(T) = 0$. Upon further analysis, it has been convenient to consider that there are the following five "target sets" for this problem:

C_1: $x_1(T) = 0$, $x_2(T) > 0$, $y(T) = 0$,

C_2: $x_1(T) = 0$ before $x_2(T) = 0$, $y(T) > 0$,

C_3: $x_1(T) = 0$ after $x_2(T) = 0$, $y(T) > 0$,

C_4: $x_1(T) > 0$, $x_2(T) = 0$, $y(T) = 0$,

C_5: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$.

The reader should note that in the above problem statement T is referred to as being undetermined. This is because T is determined by entry to one of the above five target sets. This is a function of the control applied, and hence before an allocation rule is given, it is unspecified.

3. Notation.

The symbols which are used in this appendix are defined as follows:

$A = A(R,z) = [z^2(R-1) - R]/(z-1)^2$,

$B = B(R,z) = A(z-1)^2/z^2 = [z^2(R-1) - R]/z^2$,

a_1, a_2, b_1, b_2 = constant attrition-rate coefficients,

C_i for $i = 1,2,3,4,5$ = the i^{th} part of the terminal surface as defined in section 2,

$D(C_i)$ = domain of controllability for C_i,

$g(P^o,R,z)$ = term in equation of the locus of points for which $P_1 = P_4$,

$h(P^o,R,z)$ = term in equation for boundary surface between the regions from which optimal paths lead to C_1 and C_4,

p,q,r = utilities assigned to surviving X_1, X_2 and Y forces respectively,

$p_i(t)$ for $i = 1,2,3$ = dual variable corresponding to $x_i(t)$ $(x_3(t) = y(t))$,

P_i for $i = 1,2,3,4,5$ = payoff associated with an extremal leading to C_i,

$P^o = (x_1^o, x_2^o, y_0)$ = point in the initial state space,

$Q = (-p_2(t=T))/q$

$R = a_1b_1/(a_2b_2)$,

$s = s(x_1^o, x_2^o) = b_1x_1^o + b_2x_2^o$,

t_1 = time at which X_1 is annihilated, i.e. $x_1(t_1) = 0$,

t_2 = first time at which $2b_1x_1(t_2)x_2^o + b_2(x_2^o)^2 = a_2y^2(t_2)$ for an extremal leading to C_4,

T = total time for the battle,

$v = v(\tau) = a_2p_2(\tau) - a_1p_1(\tau)$,

$$w = \cosh \sqrt{a_2 b_2}\, \tau_1(C_4) = \frac{a_1}{p_2(t=T)} \frac{(b_1 p_2(t=T) + b_2 p)}{(a_1 b_1 - a_2 b_2)},$$

x_1, x_2, y = average force strengths; with initial values x_1^o, x_2^o, y_0,

$$z = \cosh \sqrt{a_2 b_2}\, \tau_1(C_5) = \frac{a_1}{q} \frac{(b_1 q - b_2 p)}{(a_1 b_1 - a_2 b_2)} = \frac{R - \delta}{R - 1},$$

$\delta = a_1 p/(a_2 q)$,

ϕ = fraction of Y-fire directed at X_1,

τ = "backwards time" from the end of the battle defined by $= T - t$, i.e. the time remaining before the end of the battle,

$\tau_1(C_i)$ = "backwards time" of the first switch in tactics for extremals leading to C_i.

Additionally:

C_5^S refers to C_5 (the 5^{th} part of the terminal surface) which is reached by extremals with a switch in tactics,

P_5^S = payoff associated with an extremal leading to C_5^S.

4. Summary of Solution.

There are four cases to be considered

(1) $\delta \geq 1$,

(2) $R - \sqrt{R(R-1)} < \delta < 1$,

(3) $\delta = R - \sqrt{R(R-1)}$,

(4) $0 \leq \delta < R - \sqrt{R(R-1)}$,

where $\delta = a_1 p/(a_2 q)$. For Case (1): $\delta \geq 1$, the solution is given in Table I of Appendix A. Our re-esamination here changes no details of the analysis presented there.

The results of our additional analysis (which includes use of the theory of SVIC) are presented in Tables I through V. Supporting details will be presented in subsequent sections only when they differ from our previous analysis (see Appendix A).

For Case (2): $R - \sqrt{R(R-1)} < \delta < 1$, the solution is shown in Table I. Although this solution presented in Table I (Appendix F) is identical to that presented in Table II of Appendix A, there is a subtle difference in how it was obtained: our newer analysis (which invokes some new results from the theory of SVIC about the boundary conditions for the dual variables) has revealed that the domains of controllability do not "overlap" so that the extremals are unique. Hence, the extremal control turns out to be the optimal control, since it may be shown that the problem does indeed possess a solution. Previously (see Appendix A), we thought that the domain of controllability for C_4, denoted by $D(C_4)$, had an intersection of positive area (to be more precise, positive measure) with other domains of controllability.

It seems appropriate at this time to point out to the reader that the optimal control has been expressed as an open-loop control. For an open-loop control $u = u(t;x_0,t_0)$ one specifies the control as a function of time t during the length of the planning horizon $0 \leq t \leq T$. Such a control only depends on the parameter t <u>and</u> the initial conditions x_0,t_0. This control is not directly influenced by the current state of the system. On the other hand, one can consider a closed-loop (or feedback) control $u = k(x,t)$ which depends

Table I. Solution to Isbell-Marlow Problem for

$R - \sqrt{R(R-1)} < \delta < 1.$

Nonrestrictive Assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (2): $R - \sqrt{R(R-1)} < \delta < 1$ where $\delta = a_1 p/(a_2 q)$

Terminal State	Optimal Control	Domain of Controllability
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \ge s^2 + B(b_2 x_2^o)^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 > s^2 + (R-1)(b_2 x_2^o)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$	$a_1 b_1 y_0^2 \ge R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le s^2 + A(b_2 x_2^o)^2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1 b_1 y_0^2 \le Rs^2\{1 - 1/z\}$ $a_1 b_1 y_0^2 > R\{s^2 - (b_1 x_1^o)^2\}$
$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 > Rs^2\{1 - 1/z^2\}$ $a_1 b_1 y_0^2 > s + A(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 < s^2 + B(b_2 x_2^o)^2$

Definition of Times

(a) t_1 is first t such that $x_1(t_1) = 0$.

(b) t_2 is first t such that $2b_1 x_1(t_2) x_2^o + b_2 (x_2^o)^2 = a_2 y^2(t_2)$.

(c) τ_1 is determined by $\cosh \sqrt{a_2 b_2}\, \tau_1 = \dfrac{R - \delta}{R - 1}$.

upon the current state of the system. The results presented in [7] (see also [6]) are in the form of a closed-loop control. For deterministic systems (one-sided problems), it is well known that open-loop control and closed-loop control yield identical results in trajectory and payoff [4]. It is, of course, a simple matter to convert the optimal control presented, for example, in Table I into a closed-loop control. However, for optimal control of stochastic systems it is well known that open-loop and closed-loop controls do not yield the same return. In Appendix I we consider a stochastic version of (1) and consider there a closed-loop control.

It also seems appropriate to note the following for the quantities A and B which appear in inequalities defining various domains of controllability:

(a) for $0 \le \delta < R - \sqrt{R(R-1)}$,

$$\text{we have } A > B > 0, \tag{2}$$

(b) for $\delta = R - \sqrt{R(R-1)}$,

$$\text{we have } A = B = 0, \tag{3}$$

(c) for $R - \sqrt{R(R-1)} < \delta < 1$,

$$\text{we have } A < B < 0. \tag{4}$$

Furthermore, let us recall that A and B are defined by (see also Section 3).

$$A = A(R,z) = \frac{R(z^2-1) - z^2}{(z-1)^2}, \tag{5}$$

$$B = B(R,z) = \frac{R(z^2-1) - z^2}{z^2}. \tag{6}$$

The quantity z, defined by $z = \frac{R - \delta}{R - 1}$, relates A and B to δ.

For Case (3): $\delta = R - \sqrt{R(R-1)}$, the solution is shown in Table II. The reader should note the reduction in dimension (from three to two) for the domain of controllability or C_5^S, $D(C_5^S)$. Case (3) is an important case for understanding the solution to the Isbell-Marlow problem. For $R - \sqrt{R(R-1)} < \delta < 1$, both extremals and the optimal control are unique. In other words, the domains of controllability do not "overlap." For $0 \leq \delta < R - \sqrt{R(R-1)}$, extremals are no longer unique (i.e. for a point P^o in the initial state space there may be more than one extremal path leading to the terminal surface). In Case (3): $\delta = R - \sqrt{R(R-1)}$, extremals are unique for all terminal states except C_5^S. Moreover, for C_5^S, the optimal control is no longer unique: for any $P^o \in D(C_5^S)$ any policy as shown in Table II with $t_3 < t_1$ leads to the same payoff, $P_5^S = -qx_2^o\sqrt{\frac{R-1}{R}}$. Furthermore, any initial point $P^o \in D(C_1)$ or $D(C_4)$ or $D(C_5)$ with $a_1b_1y_0^2 = s^2$ also leads to exactly the same payoff when an optimal control is used, i.e. for such P^o we have $P_1 = P_4 = P_5 = P_5^S$.

For Case (4): $0 \leq \delta < R - \sqrt{R(R-1)}$, the domains of controllability and corresponding extremal controls are shown in Table III. The domains of controllability $D(C_1)$, $D(C_4)$, $D(C_5)$, and $D(C_5^S)$ all "overlap" each other (as careful study of this table will show). (Inequalities such as $A(R,z) > B(R,z)$ for $0 \leq \delta < R - \sqrt{R(R-1)}$ (which are essential in such considerations) are given in Appendix A.) Hence, step (e) of the general solution procedure presented in Appendix A must be applied. The resulting solution after this has

Table II. Solution to Isbell-Marlow Problem for

$\delta = R - \sqrt{R(R-1)}$.

Nonrestrictive assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (3): $\delta = R - \sqrt{R(R-1)}$ where $\delta = a_1 p/(a_2 q)$

Terminal State	Optimal Control	Domain of Controllability
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2)^2$ $a_1 b_1 y_0^2 \ge s^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(t) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 > s^2 + (R-1)(b_2 x_2^o)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$	$a_1 b_1 y_0^2 \ge R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le s^2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1 b_1 y_0^2 < R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le s^2$
$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_3 < t_1 \\ 0 & \text{for } t_3 < t \le T \end{cases}$	$a_1 b_1 y_0^2 = s^2$ $b_2 x_2^o > b_1 x_1^o\left(\sqrt{\frac{R}{R-1}} - 1\right)$

Definition of Times

(a) t_3 is <u>any</u> time such that $0 \le t_3 < t_1$.

(b) For t_1, t_2, and τ_1, see Table I.

Table III. Extremals for Isbell-Marlow Problem for

$0 \le \delta < R - \sqrt{R(R-1)}$

Nonrestricitve assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (4): $0 \le \delta < R - \sqrt{R(R-1)}$ where $\delta = a_1 p/(a_2 q)$

Terminal State	Extremal Control	Domain of Controllability
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \ge s^2 + B(b_2 x_2^o)^2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 > s^2 + (R-1)(b_2 x_2^o)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$\phi^*(T) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \ge R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le s^2 + A(b_2 x_2^o)^2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = 0$ for $0 \le t \le T$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \le Rs^2\{1 - 1/z^2\}$ $a_1 b_1 y_0^2 < R\{s^2 - (b_1 x_1^o)^2\}$
$C_5^S \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < t \le T \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 < Rs^2\{1 - 1/z^2\}$ $a_1 b_1 y_0^2 > s^2 + A(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 > s^2 + B(b_2 x_2^o)^2$

Definition of Times: for t_1, t_2, and τ_1, see Table I.

been done is presented in Table IV. A major difference between the results presented here and those of Appendix A is that (we have discovered since June 1971 that) extremals do lead to C_5^S. However, this gap in our earlier analysis (see Appendix A) has had no effect upon the solution which we presented earlier, since it may be shown that for an initial point $P^o \in D(C_5) \cap D(C_5^S)$ we have $P_5 \geq P_5^S$ when the corresponding extremal controls are used. Hence, extremals leading to C_5^S may be dropped (as they were previously inadvertently omitted) from further consideration.

For Case (4): $0 \leq \delta < R - \sqrt{R(R-1)}$, the solution is shown in Table IV. It is of interest to note that the solutions contain a dispersal surface (see pp. 134-141 in [5]), a solution aspect rarely encountered in optimal control problems (see [2] for the only example of which we are aware, other than those given in Isaacs' book [5]). The reader should note that the solution shown in Table IV is identical with that given earlier (see Table II of Appendix A).

A considerable amount of effort has gone into the study of extremals for $0 \leq \delta < R - \sqrt{R(R-1)}$. In Table V we show event (such as annihilation of X_1, end of battle, etc.) times and payoffs for Case (4): $0 \leq \delta < R - \sqrt{R(R-1)}$. We omit derivation of these times, which is readily done using elementary considerations (such as combination of the extremal control with the solution to the force level equations). The solution to the force level equations is consequently shown in Table VI. Additionally, the reader should note that if we have $\phi(t)$ = constant for all $t \in [t_1, t_2]$, then the following "generalized square law" holds

Table IV. Solution to Isbell-Marlow Problem for

$0 \le \delta < R - \sqrt{R(R-1)}$.

Nonrestrictive assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (4): $0 \le \delta < R - \sqrt{R(R-1)}$ where $\delta = a_1 p/(a_2 q)$

Terminal State	Region of Initial Force Levels
$C_1 \begin{cases} x_1(t_1) = 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \ge Rs^2 - R\{b_1 x_1^o[z^2(R-1)+R]/(2R) + b_2 x_2^o\}^2/z^2$ for $0 \le x_1^o < (b_2 x_2^o)/(kb_1)$ $a_1 b_1 y_0 \ge h(P^o,R,z)$ for $0 \le x_2^o \le kb_1 x_1^o/b_2$
$C_2 \begin{cases} x_1(t_1) = 0 \\ x_2(T) = 0 \\ y(T) > 0 \end{cases}$	$a_1 b_1 y_0^2 > s^2 + (R-1)(b_2 x_2^o)^2$
$C_4 \begin{cases} x_1(t_2) > 0 \\ x_2(T) = 0 \\ y(T) = 0 \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 \ge R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le h(P^o,R,z)$ for $0 \le x_2^o \le kb_1 x_1^o/b_2$
$C_5 \begin{cases} x_1(T) > 0 \\ x_2(T) > 0 \\ y(T) = 0 \end{cases}$	$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2$ $a_1 b_1 y_0^2 < R\{s^2 - (b_1 x_1^o)^2\}$ $a_1 b_1 y_0^2 \le Rs^2 - R\{b_1 x_1^o[z^2(R-1)+R]/(2R) + b_2 x_2^o\}^2/z^2$ for $0 \le x_1^o < (b_2 x_2^o)/(kb_1)$

Explanation of Symbols

(a) For t_1, t_2, and τ_1, see Table I.

(b) $k = [z^2 - R(z-1)^2]/(2R)$.

Table V. Event Times and Payoffs for Isbell-Marlow Problem when $0 \le \delta < R - R(R-1)$.

Terminal State C_1: $x_1(t_1) = 0, \quad x_2(T) > 0, \quad y(T) = 0$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s} \right\}$$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2}}{b_2 x_2^o \sqrt{R}} \right\}$$

$$P_1 = \frac{-q}{b_2 \sqrt{R}} \sqrt{s^2 + (R-1)(b_2 x_2^o)^2 - a_1 b_1 y_0^2}$$

Terminal State C_2: $x_1(t_1) = 0, \quad x_2(T) = 0, \quad y(T) > 0$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2}{\sqrt{a_1 b_1}\, y_0 - s} \right\}$$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{ \frac{b_2 x_2^o \sqrt{R}}{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2}} \right\}$$

Terminal State C_4: $x_1(t_2) = x_1(T) > 0, \quad x_2(T) = 0, \quad y(T) = 0$

t_2:

for $a_1 b_1 y_0^2 > s^2$

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{ \frac{y(t_2) - \sqrt{y^2(t_2) - y_0^2 + s^2/(a_1 b_1)}}{y_0 - s/\sqrt{a_1 b_1}} \right\}$$

for $a_1 b_1 y_0^2 < s^2$

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{ \frac{\sqrt{y^2(t_2) - y_0^2 + s^2/(a_1 b_1)} - y(t_2)}{s/\sqrt{a_1 b_1} - y_0} \right\}$$

for $a_1 b_1 y_0^2 = s$,

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{ \frac{y_0}{y(t_2)} \right\}$$

Table V. (Concluded)

where

$$y(t_2) = \sqrt{\frac{x_2^o}{a_2}\{(2R-1)b_2x_2^o + 2\sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}\}}\ ,$$

also

$$T = t_2 + \frac{1}{\sqrt{a_2b_2}}\cosh^{-1}\left\{\frac{Rb_2x_2^o + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}{(R-1)b_2x_2^o + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}\right\}$$

$$P_4 = \frac{-q\delta}{b_2R}\{b_2x_2^o(R-1) + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}\}$$

Terminal State C_5: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$

(no switch: $\phi^*(t) = 0$ for $0 \le t \le T$)

$$T = \frac{1}{\sqrt{a_2b_2}}\tanh^{-1}\left\{\frac{\sqrt{a_2b_2}\ y_0}{b_1x_1^o + b_2x_2^o}\right\}$$

Terminal State C_5^S: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$

$$\left(\text{switch in tactics:}\quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T - \tau_1 \\ 0 & \text{for } T - \tau_1 < t \le T \end{cases}\right)$$

$$\tau_1 = \frac{1}{\sqrt{a_2b_2}}\cosh^{-1}\left(\frac{R-\delta}{R-1}\right)$$

$$T = \tau_1 + \frac{1}{2\sqrt{a_1b_1}}\ln\left\{\frac{(\sqrt{R(z^2-1)} - z)(\sqrt{a_1b_1}\ y_0 + s)}{(\sqrt{a_1b_1}\ y_0 - s)(\sqrt{R(z^2-1)} + z)}\right\}$$

$$P_5^S = \frac{q}{b_2}\left\{-z(b_2x_2^o)\left(\frac{R-1}{R}\right) + \frac{1}{R}\sqrt{(z^2(R-1)-R)(a_1b_1y_0^2-s^2)}\right\}$$

Table VI. Solution to Force Level Equations for Two-on-One Combat.

Combat Equations:

$$\frac{dx_1}{dt} = -\phi a_1 y \qquad \text{with} \qquad x_1(t=0) = x_1^o,$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 y \qquad \text{with} \qquad x_2(t=0) = x_2^o,$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2 \qquad \text{with} \qquad y(t=0) = y_0.$$

Solution: when $\phi(t) = \text{constant}$ for $t \in [0,t_1]$, then

$$x_1(t) = x_1^o + \phi a_1 \left\{ \frac{(b_1 x_1^o + b_2 x_2^o)}{[\phi a_1 b_1 + (1-\phi) a_2 b_2]} \cosh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t \right.$$

$$\left. - \frac{y_0}{\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}} \sinh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t - \frac{(b_1 x_1^o + b_2 x_2^o)}{[\phi a_1 b_1 + (1-\phi) a_2 b_2]} \right\}$$

$$x_2(t) = x_2^o + (1-\phi) a_2 \left\{ \frac{(b_1 x_1^o + b_2 x_2^o)}{[\phi a_1 b_1 + (1-\phi) a_2 b_2]} \cosh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t \right.$$

$$\left. - \frac{y_0}{\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}} \sinh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t - \frac{(b_1 x_1^o + b_2 x_2^o)}{[\phi a_1 b_1 + (1-\phi) a_2 b_2]} \right\}$$

$$y(t) = y_0 \cosh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t$$

$$- \frac{(b_1 x_1^o + b_2 x_2^o)}{\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}} \sinh\sqrt{\phi a_1 b_1 + (1-\phi) a_2 b_2}\, t$$

$$\zeta^2(t=t_1) - \zeta^2(t=t_2) = \{\phi a_1 b_1 + (1-\phi) a_2 b_2\}\{y^2(t=t_1) - y^2(t=t_2)\} \quad (7)$$

where

$$\zeta(t) = b_1 x_1(t) + b_2 x_2(t).$$

This generalized square law has been used (along with the extremal control to C_i) to express $P_i = ry(T) - px_1(T) - qx_2(T)$ in the form shown in Table V.

As a final check on our theoretical developments, we had some numerical computations done for Case (4). The results shown in Table V have been used in this work. We would like to thank a M.S. thesis student, Robert L. Powers, LT, USN, for his efforts in this area. All Powers' computations have supported the solution shown in Table IV. For example, let us examine the results of computations for typical data such as that shown in Table VII. For this data, we have $\delta = 0$ and R = 2. Extremals lead to C_1, C_5, and C_5^S from P^o. For the extremal to C_1, we have $t_1 = 4.26$ minutes while $T = 18.99$ minutes and $P_1 = -217.72$. For the extremal to C_5, we have $T = 9.96$ minutes and $P_5 = -149.92$. For the extremal to C_5^S, we have $T - \tau_1 = 3.64$ minutes while $T = 16.81$ minutes and $P_5^S = -221.43$. Considering Table IV, it is readily shown that $P^o = (100,100,152)$ is such that the optimal path leads to C_5, as these numerical computations bear out. Notice that for the optimization problem (1) which we are considering and the input data shown in Table VII, making the battle last longer (by firing more at X_1) does not result in more value for survivors (from Y's standpoint). Finally, because in our earlier

Table VII. Time History of Force Levels for Extremal Leading to C_5^S

$x_1^o = 100$, $x_2^o = 100$, $y_0 = 152$

$a_1 = 0.2$ X_1 - cas./(min. × Y unit)

$a_2 = 0.1$ X_2 - cas./(min. × Y unit)

$b_1 = 0.1$ Y - cas./(min. × X_1 unit)

$b_2 = 0.2$ Y - cas./(min. × X_2 unit)

$p = 0.0$, $q = 5.0$, $r = 10.0$

time, t	$x_1(t)$	$x_2(t)$	$y(t)$
0.00	100.00	100.00	152.00
0.36	89.19	100.00	144.91
0.73	78.88	100.00	138.21
1.09	69.04	100.00	131.88
1.46	59.65	100.00	125.89
1.82	50.69	100.00	120.24
2.19	42.12	100.00	114.91
2.55	33.94	100.00	109.88
2.91	26.11	100.00	105.14
3.28	18.61	100.00	100.69
3.64	11.43	100.00	96.50
4.96	11.43	88.22	82.62
6.28	11.43	78.18	70.18
7.59	11.43	69.69	58.95
8.91	11.43	62.60	48.75
10.23	11.43	56.81	39.40
11.54	11.43	52.20	30.73
12.86	11.43	48.69	22.59
14.18	11.43	46.23	14.85
15.50	11.43	44.77	7.36
16.81	11.43	44.29	0.00

work (which is reported in Appendix A) we had erroneously concluded that no extremals led to C_5^S for $0 \leq \delta < R - \sqrt{R(R-1)}$, we also have verified that extremals do lead to C_5^S by examining the time history of the force levels computed according to the extremal control. An example of the results of such computations is shown in Table VII.

5. Necessary Conditions of Optimality.

The solution shown in Tables I, II, and IV has been developed according to the solution procedure which we outlined in Appendix A. This procedure is based upon synthesizing extremal trajectories from the basic necessary conditions of optimality. Thus, it seems appropriate to say a few words about these.

Since we have developed in Section 4.b.(1) of Appendix E the basic necessary conditions of optimality for (1), we will not repeat details again but refer the reader there for all the details. As we saw in Appendix E, the Isbell-Marlow problem requires use of results (boundary conditions for the dual variables, necessary conditions for it to be optimal to have $x_1(t) = 0$ for a finite interval of time, corner conditions) from the theory of SVIC. In the original pioneering work of Isbell and Marlow [6] these results were not used, since they had not yet been discovered and published in the mathematics literature. In our earlier work (see Appendix A in this report and also Appendix A of [8]) we had followed some heuristic reasoning of Isbell and Marlow [6] on such points.

Let us now summarize the main results of Section 4.b.(1) of Appendix E. We saw that in order for it to be optimal to have $x_1(t) = 0$

for a finite interval of time we must have

$$a_1 b_1 \geq a_2 b_2 . \tag{8}$$

At entry at t_1 to a constrained subarc on which $a_1 = 0$, we have (from the corner conditions)

$$p_1(t_1^-) = \frac{a_2}{a_1} p_2(t_1^-), \tag{9}$$

$$p_i(t_1^-) = p_i(t_1^+) \quad \text{for} \quad i = 2,3, \tag{10}$$

where t_1^- denotes a left-hand limit. Furthermore, for C_4 we have that

$$\frac{a_1 p}{a_2} < (-p_2(t{=}T)) \leq q. \tag{11}$$

The latter of these two inequalities was conjectured in Appendix A (in June 1971) to be a property of an extremal. Our recent work using the theory of SVIC has confirmed this conjecture.

6. Existence of Extremals Leading to C_5^S.

In our earlier work [9] we erroneously concluded that for $0 \leq \delta < R - \sqrt{R(R-1)}$ there were no extremals leading to C_5^S. Let us now retrace our earlier development in a correct manner.

There are two subcases for entry to C_5. We consider the extremals for which

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq T - \tau_1, \\ 0 & \text{for} \quad T - \tau_1 < t \leq T. \end{cases} \tag{12}$$

For convenience, let us say that such an extremal leads to C_5^S. Combining the above extremal control with the generalized square law (7), we obtain

$$R(b_2x_2(T))^2 + 2Rb_1b_2x_1(T-\tau_1)x_2(T) + (b_1x_1(T-\tau_1))^2$$

$$+ 2(1-R)b_1b_2x_1(T-\tau_1)x_2^o + a_1b_1y_0^2 - s^2 + (1-R)(b_2x_2^o)^2 = 0. \quad (13)$$

A backwards integration of the state equations of $0 \leq \tau \leq \tau_1$ with $x_2(\tau=0) = x_2(T)$ and $y(\tau=0) = 0$ yields that

$$b_2x_2^o = b_2x_2(T)z + b_1x_1(T-\tau_1)(z-1), \quad (14)$$

since $x_2(t=T-\tau_1) = x_2^o$. It should be noted that in deriving (14) we have assumed that $x_1(T-\tau_1) > 0$. (13) and (14) may be combined to yield

$$(b_2x_2(T)-b_2x_2^o)^2 = (a_1b_1y_0^2-s^2)/A, \quad (15)$$

and hence we require that the right-hand side of (15) be positive in order that C_5^S be reached. Thus, for $P^o \in D(C_5^S)$ we must have

$$a_1b_1y_0^2 < s^2 \quad \text{for} \quad A < 0, \quad (16)$$

$$a_1b_1y_0^2 = s^2 \quad \text{for} \quad A = 0, \quad (17)$$

$$a_1b_1y_0^2 > s^2 \quad \text{for} \quad A > 0, \quad (18)$$

where the reader should keep (2) through (4) in mind.

Now when $A < 0$ (corresponding to $R - \sqrt{R(R-1)} < \delta < 1$), (15) may be solved to yield

$$b_2x_2(T) = b_2x_2^o + (1-z)\sqrt{\frac{s^2 - a_1b_1y_0^2}{R - z^2(R-1)}} \; . \qquad (19)$$

The requirement that $x_2(T) > 0$ then leads to

$$a_1b_1y_0^2 > s^2 + A(b_2x_2^o)^2, \qquad (20)$$

where we have made use of (5). Combination of (14) and (19) yields that

$$b_1x_1(T-\tau_1) = -b_2x_2^o + z\sqrt{\frac{s^2 - a_1b_1y_0^2}{R - z^2(R-1)}} \; . \qquad (21)$$

The requirement that $x_1(T-\tau_1) > 0$ then leads to

$$a_1b_1y_0^2 < s^2 + B(b_2x_2^o)^2, \qquad (22)$$

where we have made use of (6). Furthermore, the time $T - \tau_1$ may be determined by solving the following equation (see Table VI)

$y(T-\tau_1) = y_0 \cosh\sqrt{a_1b_1}\,(T-\tau_1) - \dfrac{s}{\sqrt{a_1b_1}} \sinh\sqrt{a_1b_1}\,(T-\tau_1)$ to yield

$$T - \tau_1 = \frac{1}{\sqrt{a_1b_1}}\, \ell n \; \frac{\sqrt{a_1b_1y_2(T-\tau_1) - a_1b_1y_0^2 + s^2} - \sqrt{a_1b_1}\; y(T-\tau_1)}{s - \sqrt{a_1b_1}\; y_0} \qquad (23)$$

The requirement that the quantity in brackets is greater than one leads to

$$y_0 > y(T-\tau_1). \qquad (24)$$

Now, the generalized square law (7) may be combined with the extremal control (12) and (14) to yield

$$\sqrt{a_1 b_1}\ y(T-\tau_1) = \sqrt{R(z^2-1)}\ \sqrt{\frac{s^2 - a_1 b_1 y_0^2}{R - z^2(R-1)}}\ . \tag{25}$$

(24) and (25) may be combined to yield

$$a_1 b_1 y_0^2 > Rs^2\ \{1 - 1/z^2\}. \tag{26}$$

Furthermore, (23) and (25) may be combined to yield

$$T - \tau_1 = \frac{1}{2\sqrt{a_1 b_1}}\ \ell n \left\{\frac{(z-\sqrt{R(z^2-1)})(s+\sqrt{a_1 b_1}\ y_0)}{(s-\sqrt{a_1 b_1}\ y_0)(z+\sqrt{R(z^2-1)})}\right\}. \tag{27}$$

Finally, (19) may be used to show that the payoff P_5^S corresponding to use of the extremal control (12) is given by

$$P_5^S = \frac{q}{b_2}\left\{-z\left(\frac{R-1}{R}\right)b_2 x_2^o + \frac{1}{R}\sqrt{(R-z^2(R-1))(s^2-a_1 b_1 y_0^2)}\right\}\ . \tag{28}$$

When $A > 0$ (corresponding to $0 \le \delta < R - \sqrt{R(R-1)}$), we must have $a_1 b_1 y_0^2 > s^2$ by (18) so that analysis similar to the above readily yields results similar to (20), (22), and (26) but with the inequalities reversed (see Table III). In our previous work, we had erroneously concluded that this case was impossible.

Next, we consider the extremals for which

$$\phi^*(t) = 0 \quad \text{for} \quad 0 \le t \le T. \tag{29}$$

Let us adopt the terminology that such an extremal leads to C_5 with payoff P_5. We omit discussion of results previously derived in Appendix A. In that appendix we showed that

$$b_2x_2(T) = - b_1x_1^o + \sqrt{s^2 - a_1b_1y_0^2/R}. \tag{30}$$

Using (30), we may express P_5 in the following form

$$P_5 = \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1x_1^o - \sqrt{s^2 - a_1b_1y_0^2/R}\right\}. \tag{31}$$

7. <u>Proof of a Dominance Theorem</u>.

In this section we show that for $0 \le \delta < R - \sqrt{R(R-1)}$ we need not consider extremals that lead to C_5^S, since the payoff associated with such an extremal path is exceeded (or dominated) by the payoff associated with an extremal leading to C_5. Hence, our previous oversight on extremals leading to C_5^S is of no consequence.

Our main result is stated below as Theorem 1. It is convenient, however, to first explain our terminology in a precise fashion. To be precise, then, we have

Definition 1: We say that an extremal leads to C_5^S when we use the extremal control

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T - \tau_1, \\ 0 & \text{for } T - \tau_1 < t \le T, \end{cases}$$

and reach the terminal state $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$. (τ_1 is given in Table V.)

We denote the payoff associated with an extremal leading to C_5^S by P_5^S. We have that for $0 \le \delta < R - \sqrt{R(R-1)}$

$$P_5^S = \frac{q}{b_2}\left\{-z\left(\frac{R-1}{R}\right)b_2x_2^o + \frac{1}{R}\sqrt{(z^2(R-1)-R)(a_1b_1y_0-s^2)}\right\}. \tag{32}$$

Furthermore, the domain of controllability for C_5^S, denoted by $D(C_5^S)$, is given by

for $0 \le \delta < R - \sqrt{R(R-1)}$

$$D(C_5^S) = \{P^o = (x_1^o, x_2^o, y_0) \mid a_1 b_1 y_0^2 \le Rs^2(1-1/z^2) \quad \text{and}$$

$$s^2 < s^2 + B(b_2 x_2^o)^2 < a_1 b_1 y_0^2 < s^2 + A(b_2 x_2^o)^2\}. \tag{33}$$

We also have

Definition 2: We say that an extremal leads to C_5 when we use the extremal control

$$\phi^*(t) = 0 \quad \text{for} \quad 0 \le t \le T,$$

and reach the terminal state $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$.

We denote the payoff associated with an extremal leading to C_5 by P_5. We have that for $0 \le \delta < R - \sqrt{R(R-1)}$

$$P_5 = \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1 x_1^o - \sqrt{s^2 - a_1 b_1 y_0^2/R}\right\}. \tag{34}$$

Furthermore, the domain of controllability for C_5, denoted by $D(C_5)$, is given by

for $0 \le \delta < R - \sqrt{R(R-1)}$

$$D(C_5) = \{P^o = (x_1^o, x_2^o, y_0) \mid a_1 b_1 y_0^2 \le Rs^2(1-1/z^2) \quad \text{and}$$

$$a_1 b_1 y_0^2 < R\{s^2 - (b_1 x_1^o)^2\}\}. \tag{35}$$

To be precise, we must also add the following condition to (33) and (35)

$$a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^o)^2. \tag{36}$$

Since (36) does not enter into the proof of Theorem 1, we have (for convenience) omitted it.

We now state the main result of this section as Theorem 1.

THEOREM 1: Assume that $0 \leq \delta < R - \sqrt{R(R-1)}$. (This is equivalent to $z > \sqrt{R/(R-1)}$.) Then $P^o = (x_1^o, x_2^o, y_0) \in D(C_5) \cap D(C_5^S)$ implies that $P_5 \geq P_5^S$ with equality holding only when $a_1b_1y_0^2 = Rs^2(1-1/z^2)$.

PROOF:

A. First we will show that

$$s^2 < a_1b_1y_0^2 < Rs^2(1-1/z^2), \tag{37}$$

and

$$\frac{R}{R-1} < z^2, \tag{38}$$

imply that $P_5^S < P_5$. For $P^o \in D(C_5) \cap D(C_5^S)$ we have (postponing for now the case when equality holds)

$$a_1b_1y_0^2 < Rs^2(1-1/z^2), \tag{39}$$

which implies that

$$R(z^2-1)s^2 - z^2a_1b_1y_0^2 > 0. \tag{40}$$

Squaring (38), we may rearrange the result to obtain

$$(2Rsz\sqrt{s^2 - a_1b_1y_0/R})^2 < \{R(z^2+1)s^2 - z^2a_1b_1y_0^2\}^2. \tag{41}$$

By observing that $1 - 1/z^2 < 1 + 1/z^2$ and considering (39), it is readily seen that

$$R(z^2+1)s^2 - z^2a_1b_1y_0^2 > 0, \tag{42}$$

so that we may take square roots in (41) to obtain

$$2Rsz\sqrt{s^2 - a_1b_1y_0^2/R} < R(z^2+1)s^2 - z^2a_1b_1y_0^2. \tag{43}$$

Now it is easily seen that (41) may be rearranged to yield

$$(z^2(R-1)-R)(a_1b_1y_0^2-s^2) < \{z(R-1)s - R\sqrt{s^2 - a_1b_1y_0/R}\}^2. \tag{44}$$

The conditions $s^2 < a_1b_1y_0^2$ and $z^2 > R/(R-1)$ are readily combined to yield

$$R^2(s^2-a_1b_1y_0^2/R) < z^2(R-1)^2s^2. \tag{45}$$

Recalling that for $P^o \in D(C_5) \cap D(C_5^S)$ we have $a_1b_1y_0 < Rs^2(1-1/z^2) < Rs$, we may extract square roots in (45) to obtain

$$z(R-1)s - R\sqrt{s^2 - a_1b_1y_0^2/R} > 0. \tag{46}$$

Considering (37), (38), and (46), we may extract square roots in (44) to obtain

$$\frac{1}{R}\sqrt{(z^2(R-1)-R)(a_1b_1y_0^2-s^2)} < z\left(\frac{R-1}{R}\right)(b_1x_1^o+b_2x_2^o) - \sqrt{s^2 - a_1b_1y_0^2/R}. \tag{47}$$

Some additional manipulations on (47) now readily yield that (37) and (38) imply that $P_5 > P_5^S$.

B. Next we show that

$$a_1 b_1 y_0 = Rs^2(1-1/z^2), \tag{48}$$

implies that $P_5 = P_5^S$. It is readily shown that (48) leads to

$$\sqrt{(z^2(R-1)-R)(a_1 b_1 y_0^2 - s\ )} = \frac{s^2}{z}(z^2(R-1)-R). \tag{49}$$

Now (32) and (49) may be combined to yield

$$P_5^S = \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1 x_1^o - \frac{s}{z}\right\}. \tag{50}$$

Furthermore, (48) also leads to

$$s^2 - a_1 b_1 y_0^2/R = s^2/z^2, \tag{51}$$

which may be combined with (34) to yield

$$P_5 = \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1 x_1^o - \frac{s}{z}\right\}. \tag{52}$$

Consideration of (50) and (52) shows that (48) implies that $P_5 = P_5^S$.

By A. and B. above, Theorem 1 is proved. Q.E.D.

8. Development of Conditions for Extremals to Reach C_4.

Another topic for which our previous discussion [9] was inadequate was the development of conditions for extremals to reach C_4. (Previously, we heuristically developed (66) and had not shown that (70) was a necessary condition.)

Let us define

$$w = w(p_2(t=T)) = \frac{a_1(b_1p_2(t=T)+b_2p)}{p_2(t=T)(a_1b_1-a_2b_2)} ,$$

which we may write as

$$w = w(Q) = \frac{1}{(R-1)}\left\{R - \frac{\delta}{Q}\right\}, \tag{53}$$

where

$$Q = (-p_2(t=T))/q. \tag{54}$$

Then recalling (11) (which was a necessary condition of optimality), we have that

for $\delta < Q \leq 1$,

we have $$1 < w(Q) \leq \frac{R-\delta}{R-1} = z, \tag{55}$$

where $w(Q)$ is a strictly increasing function of Q and $w(Q=\delta) = 1$.

Now

$$w = \cosh\sqrt{a_2b_2}\,(T-t_2), \tag{56}$$

and

$$z = \cosh\sqrt{a_2b_2}\,\tau_1 = \cosh\sqrt{a_2b_2}\,\tau_1(C_5^S).$$

Since $\cosh x$ is a strictly increasing function, (55) and (56) readily yield

$$T - t_2 \leq \tau_1 = \tau_1(C_5^S). \tag{57}$$

Now, the extremal control is

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_2, \\ 0 & \text{for } t_2 \leq t \leq T, \end{cases} \tag{58}$$

where t_2 is the smallest t such that

$$a_1b_1y^2(t_2) = R\{(b_1x_1(t_2)+b_2x_2^o)^2 - (b_1x_1(t_2))^2\}. \tag{59}$$

In [9] we showed that the extremal control (58) and the generalized square law (7) yield

$$b_1x_1(t_2) = b_2x_2^o(R-1) + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}\ . \tag{60}$$

Clearly we must have $b_1x_1(t_2) \leq b_1x_1^o$, and this may be combined with (60) to yield

$$b_2x_2^o(R-1) \leq b_1x_1^o. \tag{61}$$

A backwards integration of the state equations using (58) yields for $t_2 \leq t \leq T$

$$b_2x_2(t) = b_1x_1(t_2)\{\cosh\sqrt{a_2b_2}\ (T-t) - 1\}, \tag{62}$$

so that

$$b_2x_2(t_2) = b_2x_2^o = b_1x_1(t_2)\{\cosh\sqrt{a_2b_2}\ (T-t_2) - 1\}. \tag{63}$$

Then (60) and (63) may be combined to yield

$$T - t_2 = \frac{1}{\sqrt{a_2b_2}} \cosh^{-1}\left\{\frac{Rb_2x_2^o + \sqrt{s^2 + R(R-1)(B_2x_2^o)^2 - a_1b_1y_0^2}}{(R-1)b_2x_2^o + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}\right\}. \tag{64}$$

Recalling that

$$\tau_1 = \tau_1(C_5^S) = \frac{1}{\sqrt{a_2b_2}} \cosh^{-1}\left(\frac{R-\delta}{R-1}\right)\ , \tag{65}$$

we may combine (57) and (64) to obtain

$$a_1 b_1 y_0 \leq s \; + A(b_2 x_2^o)^2. \tag{66}$$

Now, since $\phi^*(t) = 1$ for $0 \leq t \leq t_2$, we have by the generalized square law (7)

$$a_1 b_1 y_0^2 - a_1 b_1 y^2(t_2) = s^2 - (b_1 x_1(t_2) + b_2 x_2^o)^2. \tag{67}$$

We may combine (59) and (67) to obtain

$$a_1 b_1 y_0^2 = (R-1)\{(b_1 x_1(t_2) + b_2 x_2^o)^2 - s^2\} + R\{(b_1 x_1^o)^2 - (b_1 x_1(t_2))^2\}$$
$$+ R\{s^2 - (b_1 x_1^o)^2\}. \tag{68}$$

We will prove as Proposition 1 that

$$R\{(b_1 x_1^o)^2 - (b_1 x_1(t_2))^2\} - (R-1)\{s^2 - (b_1 x_1(t_2) + b_2 x_2^o)^2\} \geq 0, \tag{69}$$

so that (68) and (69) yield

$$a_1 b_1 y_0^2 \geq R\{s^2 - (b_1 x_1^o)^2\}. \tag{70}$$

It remains to prove the following proposition

PROPOSITION 1: A necessary and (with appropriate additional assumptions) sufficient condition for (69) to be true is that $b_2 x_2^o(R-1) \leq b_1 x_1^o$.

PROOF: We prove necessity only (sufficiency follows by reversing the chain of arguments).

It is readily shown that (69) may be rearranged to yield

$$(b_1x_1^o)^2 - (b_1x_1(t_2))^2 \geq 2(R-1)b_2x_2^o\{b_1x_1^o - b_1x_1(t_2)\}. \qquad (71)$$

Now (71) may be factored and (assuming that $b_1x_1^o > b_1x_1(t_2)$) a common term cancelled on both sides to yield

$$b_1x_1^o + b_1x_1(t_2) \geq 2(R-1)b_2x_2^o \qquad (72)$$

Combining (60) with (72), we obtain

$$\sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2} \geq b_2x_2^o(R-1) - b_1x_1^o, \qquad (73)$$

and the necessity of the proposition is proved. Q.E.D.

The time t_2 in (58) may be explicitly determined by solving the equation (see Table VI)

$$y(t=t_2) = y_0 \cosh\sqrt{a_1b_1}\, t_2 - \frac{s}{\sqrt{a_1b_1}} \sinh\sqrt{a_1b_1}\, t_2, \qquad (74)$$

and this yields

$$t_2 = \frac{1}{\sqrt{a_1b_1}} \ln\left\{\frac{y(t_2) - \sqrt{y^2(t_2) - y_0^2 + s^2/(a_1b_1)}}{y_0 - s/\sqrt{a_1b_1}}\right\} \qquad (75)$$

Since the quantity in brackets must be ≥ 1 (for all cases, i.e. $a_1b_1y_0^2 >, =, < s^2$), we must have

$$y_0 \geq y(t_2) > 0. \qquad (76)$$

Now (59) and (60) may be combined to yield

$$y^2(t_2) = \frac{x_2^o}{a_2}\{(2R-1)b_2x_2^o + 2\sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}\} , \tag{77}$$

so that when this is combined with (76), we obtain

$$\left\{\frac{a_2b_2y_0^2}{b_2x_2^o} - (2R-1)b_2x_2^o\right\} \geq 2\sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2} . \tag{78}$$

Hence, we must have

$$a_1b_1y_0^2 \geq R(2R-1)(b_2x_2^o)^2. \tag{79}$$

If (79) holds, then squaring both sides of (78) again leads to (70).

Moreover, (61) and (70) yield (79) as Proposition 2 shows.

PROPOSITION 2: Assume that

$$(R-1)b_2x_2^o \leq b_1x_1^o.$$

Then

$$a_1b_1y_0^2 \geq R\{s^2 - (b_1x_1^o)^2\} \text{ implies that}$$

$$a_1b_1y_0^2 \geq R(2R-1)(b_2x_2^o)^2.$$

PROOF: We may manipulate $(R-1)b_2x_2^o \leq b_1x_1^o$ to obtain

$$(2R-1)(b_2x_2^o)^2 \leq 2b_1x_1^ob_2x_2^o + (b_2x_2^o)^2.$$

Hence

$$R(2R-1)(b_2x_2^o)^2 \leq R\{s^2 - (b_1x_1^o)^2\},$$

and whence the proposition. Q.E.D.

9. Analysis of Dispersal Surface.

For $0 \le \delta < R - \sqrt{R(R-1)}$, the domains of controllability for all the terminal states are shown in Table III. It is easily seen that many of these domains overlap each other (the inequalities (47) through (53) of Appendix A are useful in such considerations). Hence, we must use "considerations in the large" to select the optimal policy from among the candidate extremal policies. This is essentially step (e) of our general solution procedure outlined in Section 4. of Appendix A.

Our "considerations in the large" have led us to discover a new aspect of the Isbell-Marlow problem [6]: the solution contains dispersal surfaces for $0 \le \delta < R - \sqrt{R(R-1)}$. A dispersal surface (see pp. 134-141 in [5]) is a surface only away from which optimal paths lead. It is the locus of points such that the payoffs from two families of extremals are equal.

Let us now show how the dispersal surface which separates optimal paths leading to C_1 and also C_5 is explicitly determined. We have previously summarized results in [9] (see Appendix A in this report), but we present the details here.

It seems appropriate to firmly establish our notation first, however. To be precise, then, we have

Definition 3: We say that an extremal leads to C_1 when we use the extremal control

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le t_1, \\ 0 & \text{for} \quad t_1 < t \le T, \end{cases}$$

and reach the terminal state $x_1(T) = x_1(t_1) = 0$, $x_2(T) > 0$, $y(T) = 0$.

We denote the payoff associated with an extremal leading to C_1 by P_1. We have that

$$P_1 = \frac{-q}{b_2\sqrt{R}} \sqrt{s^2 + (R-1)(b_2x_2^o)^2 - a_1b_1y_0^2} \, . \qquad (80)$$

Furthermore, the domain of controllability for C_1, denoted by $D(C_1)$, is given by

$$D(C_1) = \{P^o = (x_1^o, x_2^o, y_0) \mid a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^o)^2$$

$$\text{and} \qquad a_1b_1y_0^2 \geq s^2 + B(b_2x_2^o)^2\}. \qquad (81)$$

We refer the reader to Section 7 for the corresponding notation and results for C_5^S and C_5. However, let us be more precise in (35); we now write this as

for $0 \leq \delta < R - \sqrt{R(R-1)}$

$$D(C_5) = \{P^o = (x_1^o, x_2^o, y_0) \left| \begin{array}{l} a_1b_1y_0^2 < s^2 + (R-1)(b_2x_2^o)^2, \\[2ex] a_1b_1y_0^2 \leq Rs^2(1-1/z^2), \text{ and} \\[2ex] a_1b_1y_0^2 < R\{s^2 - (b_1x_1^o)^2\}.\} \end{array} \right. . \qquad (82)$$

Considering (33), (35) (or equivalently (82), and (81), we see that $D(C_1) \cap D(C_5) \cap D(C_5^S)$ is nonempty (see inequalities (47) through (53) of Appendix A). However, Theorem 1 tells us that we need not consider further extremals leading to C_5^S. Thus, we must determine optimal paths from extremals for $P^o \in D(C_1) \cap D(C_5)$.

Now by (81) and (82) P^o such that $a_1b_1y_0^2 = s^2 + B(b_2x_2)^2$ clearly belongs to $D(C_1) \cap D(C_5)$. (The reader should recall (2): $B > 0$.) We begin by stating and proving Theorem 2.

THEOREM 2: Assume that $0 \leq \delta < R - \sqrt{R(R-1)}$. Then for $a_1 b_1 y_0^2 = s^2 + B(b_2 x_2^o)^2$, we have $P_5 > P_1$.

PROOF:

A. Let us first observe that $\delta < R - \sqrt{R(R-1)}$ implies that

$$z^2 > \left(\frac{R}{R-1}\right). \tag{83}$$

B. Now it is readily shown that (83) leads to

$$z^2\left(\frac{R-1}{R}\right)(b_1 x_1^o)^2 + 2b_1 x_1^o b_2 x_2^o > s^2 - (b_2 x_2^o)^2. \tag{84}$$

We may add the term $(b_2 x_2^o)^2/z^2$ to both sides of (84). This makes the left-hand-side of (84) a perfect square so that extracting square roots, we obtain

$$z\left(\frac{R-1}{R}\right)b_1 x_1^o + \frac{b_2 x_2^o}{z} > \sqrt{s^2 - \frac{1}{R}\{s^2 + B(b_2 x_2^o)^2\}}, \tag{85}$$

where we have made use of the definition (6) of B. Recalling that we are considering the case when $a_1 b_1 y_0^2 = s^2 + B(b_2 x_2)^2$, we see that (85) leads to

$$z\left(\frac{R-1}{R}\right)b_1 x_1^o - \sqrt{s^2 - a_1 b_1 y_0^2/R} > -\frac{b_2 x_2^o}{z}. \tag{86}$$

Hence, considering (34) for P_5, we have

$$P_5 > -\frac{q}{b_2}\frac{(b_2 x_2^o)}{z}. \tag{87}$$

C. For $a_1 b_1 y_0^2 = s^2 + B(b_2 x_2)^2$, it is readily shown by (80) that P_1 reduces to

$$P_1 = -\frac{q}{b_2}\frac{(b_2 x_2^o)}{z}, \tag{88}$$

whence the theorem follows. Q.E.D.

Now for $0 \le \delta < R - \sqrt{R(R-1)}$, we showed in Appendix A that

$$s^2 + B(b_2 x_2^o)^2 < Rs^2(1-1/z^2). \tag{89}$$

Hence, P^o such that $a_1 b_1 y_0^2 = Rs^2(1-1/z^2)$ clearly belongs to $D(C_1) \cap D(C_5)$. We now state and prove Theorem 3.

THEOREM 3: Assume that $0 \le \delta < R - \sqrt{R(R-1)}$. Then for $a_1 b_1 y_0^2 = Rs^2(1-1/z^2)$ and $P^o \in D(C_1) \cap D(C_5)$, we have $P_1 > P_5$.

PROOF:

A. Again we observe that $\delta < R - \sqrt{R(R-1)}$ implies that

$$z^2 > \left(\frac{R}{R-1}\right). \tag{90}$$

B. We may readily obtain from (90) that

$$z^2\left(\frac{R-1}{R}\right)(b_1 x_1^o)^2 > (s-b_2 x_2^o)^2, \tag{91}$$

which is easily further manipulated upon to yield

$$\left\{\frac{s}{z} - z\left(\frac{R-1}{R}\right)b_1 x_1^o\right\}^2 > \frac{1}{R}\{s^2 + (R-1)(b_2 x_2^o)^2 - Rs^2(1-1/z^2)\}. \tag{92}$$

Now for $\delta \ge 0$, we readily have

$$b_1x_1^o \geq z\left(\frac{R-1}{R}\right)b_1x_1^o. \tag{93}$$

By (82) we have that for $P^o \in D(C_5)$

$$a_1b_1y_0^2 < R\{s^2 - (b_1x_1)^2\}. \tag{94}$$

Since we are assuming that

$$a_1b_1y_0^2 = Rs^2(1-1/z^2), \tag{95}$$

we may obtain from (94) that

$$b_1x_1^o < \frac{s}{z}, \tag{96}$$

so that (93) and (96) yield that

$$\frac{s}{z} > z\left(\frac{R-1}{R}\right)b_1x_1^o. \tag{97}$$

Considering (97), we may extract square roots in (92) to obtain

$$\frac{s}{z} = z\left(\frac{R-1}{R}\right)b_1x_1^o > \frac{1}{\sqrt{R}}\sqrt{s^2 + (R-1)(b_2x_2^o)^2 - Rs\ (1-1/z^2)}. \tag{98}$$

Recalling the expression (80) for P_1, we readily see from (98) that

$$P_1 > \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1x_1^o - \frac{s}{z}\right\}, \tag{99}$$

C. Since we are assuming that (95) holds, it is readily shown using (34) that P_5 reduces to

$$P_5 = \frac{q}{b_2}\left\{z\left(\frac{R-1}{R}\right)b_1x_1^o - \frac{s}{z}\right\}, \tag{100}$$

whence the theorem follows. Q.E.D.

Thus, by Theorems 2 and 3 we have shown that for $0 \leq \delta < R - \sqrt{R(R-1)}$ (or, equivalently, for $\sqrt{R/(R-1)} < z \leq R/(R-1)$)

A. $P_5 > P_1$ for $a_1b_1y_0^2 = s^2 + B(b_2x_2^o)^2$,

or

$$(P_1 - P_5) < 0$$

and

B. $P_1 > P_5$ for $a_1b_1y_0^2 = Rs^2(1-1/z^2)$,

or

$$(P_1 - P_5) > 0$$

where we also assume that $P^o \in D(C_1) \cap D(C_5)$. We also note that $s^2 + B(b_2x_2^o)^2 < Rs^2(1-1/z^2)$ for $0 \leq \delta < R - \sqrt{R(R-1)}$. The locus of points for which $P_1 = P_5$ (or $P_1 - P_5 = 0$) is the <u>dispersal surface</u>. It is uniquely determined, since we always have

$$\frac{\partial}{\partial y_0}(P_1 - P_5) > 0. \tag{101}$$

We now prove (101) as Theorem 4.

THEOREM 4: Assume that $x_1^o > 0$ and $R > 1$. Then

$$\frac{\partial}{\partial y_0}(P_1 - P_5) > 0.$$

PROOF:

A. Using (34) and (80), it is easily computed that

$$\frac{\partial}{\partial y_0}(P_1 - P_5) = \frac{\left[\dfrac{a_1b_1y_0q}{b_2\sqrt{R}}\right]\left\{1 - \sqrt{\dfrac{s^2 + (R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}{Rs^2 - a_1b_1y_0^2}}\right\}}{\sqrt{s^2 + (R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}. \tag{102}$$

B. Now for $x_1^o > 0$, we have

$$(R-1)(b_2x_2^o)^2 < (R-1)s^2 \quad \text{for} \quad R > 1. \tag{103}$$

We may rearrange (103) and perform further manipulations to obtain

$$\sqrt{\frac{s^2 + (R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}{Rs^2 - a_1b_1y_0^2}} < 1, \tag{104}$$

whence the theorem follows. Q.E.D.

Our results (Theorems 2, 3, and 4) are summarized in Table VIII. The dispersal surface is the locus of points such that $P_1 = P_5$. Equating (34) and (80), we find the equation of the dispersal surface $S_{D_{1-5}}$ as

$$a_1b_1y_0^2 = Rs^2 - R\{b_1x_1^o[z^2(R-1) + R]/(2R) + b_2x_2^o\}^2/z^2, \tag{105}$$

for

$$0 \le x_1^o < 2Rb_2x_2^o/\{b_1[z^2 - R(z-1)^2]\}. \tag{106}$$

Expression (106) must also be considered, since (94) must also hold for extremals leading to C_5. Considering Theorem 4 and (105) (with (106)), we readily obtain the corresponding results shown in Table IV.

For $0 \le \delta < R - \sqrt{R(R-1)}$, there may also be a dispersal surface separating optimal paths which lead to C_1 and C_4. As noted in Appendix A, the details are much messier than those for $S_{D_{1-5}}$, and an explicit expression has not been obtained for $S_{D_{1-4}}$.

Table VIII. Summary of Results on Dispersal Surface which Separates Optimal Paths Leading to C_1 and C_5.

For $0 \leq \delta < R - \sqrt{R(R-1)}$ (i.e. for $\sqrt{\frac{R}{R-1}} < z \leq \frac{R}{R-1}$)

1. $(P_1 - P_5) < 0$ for $a_1 b_1 y_0^2 = s^2 + B(b_2 x_2^o)^2$

2. $(P_1 - P_5) > 0$ for $a_1 b_1 y_0^2 = Rs^2(1 - 1/z^2)$

3. $\frac{\partial}{\partial y_0}(P_1 - P_5) > 0$ for $x_1^o > 0$ and $R > 1$

Let us note that $s^2 + B(b_2 x_2^o)^2 < Rs^2(1 - 1/z^2)$.

Then 1, 2, and 3 imply that there exists a uniquely determined surface such that $P_1 = P_5$. Denote this surface as $S_{D_{1-5}}$.

Clearly $S_{D_{1-5}} \subset \{D(C_1) \cap D(C_5)\}$.

10. Proof of a Useful Inequality.

The following lemma has been useful in developing the domains of controllability for C_1 and C_5.

LEMMA 1: Assume that $0 \le y \le 1$ and $x \ge 1$. Then a necessary and sufficient condition for

$\tanh^{-1} y \ge \cosh^{-1} x$ to hold is that

$$y^2 \ge \frac{x^2-1}{x^2} .$$

PROOF: We recall that

$$\tanh^{-1} y = \frac{1}{2} \ln\left(\frac{1+y}{1-y}\right) , \tag{107}$$

and

$$\cosh^{-1} x = \ln(x+\sqrt{x^2-1}). \tag{108}$$

A. Proof of necessity:

Assuming that $\tanh^{-1} y \ge \cosh^{-1} x$, we may use (107) and (108) to obtain

$$\ln\left(\frac{1+y}{1-y}\right) \ge \ln(x+\sqrt{x^2-1})^2. \tag{109}$$

Since the logarighm is a strictly increasing function, (109) yields that

$$1 + \frac{1+y}{1-y} \ge 2x^2 + 2x\sqrt{x^2-1}. \tag{110}$$

Further manipulation on (110) lead to

$$\frac{1}{1-y} - x^2 \ge x\sqrt{x^2-1}. \tag{111}$$

Since the right-hand-side of (111) is ≥ 0, we may square both sides to obtain

$$\frac{1}{(1-y)^2} - \frac{2x^2}{1-y} \geq -x^2. \tag{112}$$

Now since $1 - y \geq 0$, we may multiply both sides of (112) without changing the inequality. Rearranging the resulting expression, we readily obtain

$$y^2 \geq \frac{x^2-1}{x^2} \, . \tag{113}$$

Thus, $\tanh^{-1} y \geq \cosh^{-1} x$ implies that (113) holds.

B. Proof of sufficiency:

Assuming that $y \geq \frac{x^2-1}{x^2}$ and recalling that $0 \leq y \leq 1$, we readily see that

$$y \geq y^2 \geq \frac{x^2-1}{x^2} \, . \tag{114}$$

Writing this as

$$y^2 \geq 1 - \frac{1}{x^2} \, , \tag{115}$$

a straightforward chain of manipulations leads us to

$$\left(\frac{1}{1-y} - x^2\right) \geq \{x\sqrt{x^2-1}\}^2. \tag{116}$$

Combining (114) and (115), we see that

$$\frac{1}{1-y} - x^2 \geq 0, \tag{117}$$

so that we may extract square roots in (116) to obtain

$$\frac{1}{1-y} - x^2 \geq x\sqrt{x^2-1}. \tag{118}$$

Straightforward manipulations upon (118) lead us to

$$\sqrt{\frac{1+y}{1-y}} \geq x + \sqrt{x^2-1}, \tag{119}$$

whence by (107) and (108) we obtain

$$\tanh^{-1} y \geq \cosh^{-1} x. \tag{120}$$

Thus, $y^2 \geq \frac{x^2-1}{x^2}$ implies that (120) holds. Q.E.D.

Let us now illustrate the use of Lemma 1. Consideration of (29) yields that $T = T(C_5)$ is given by

$$T = \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1} \left\{ \frac{\sqrt{a_2 b_2}\ y_0}{s} \right\}. \tag{121}$$

We also have that $\tau_1 = \tau_1(C_5^S)$ is given by

$$\tau_1 = \frac{1}{\sqrt{a_2 b_2}} \cosh^{-1} z. \tag{122}$$

Now C_5 is reached from P^o when $T(C_5) \leq \tau_1(C_5^S)$, since in both cases we use $\phi^*(t) = 0$ for the final portion of the battle. Applying Lemma 1, we readily find that

$$a_1 b_1 y_0^2 \leq Rs\ (1-1/z^2). \tag{123}$$

Lemma 1 is also used to determine the condition

$$a_1 b_1 y_0^2 \geq s^2 + B(b_2 x_2^o)^2, \tag{124}$$

for $D(C_1)$.

11. Discussion.

For a discussion of the structure of the optimal allocation policies, the reader is directed to Section 6 of Appendix A. Here we will try to put into perspective what we have learned from this, our latest, examination of the Isbell-Marlow problem.

First of all, we have concluded that the theory of SVIC is absolutely essential for solving such problems. Many of the results in the pioneering 1956 work of Isbell and Marlow [6] (see also [7]) are unsupported by such analysis (which uses theory developed since 1956). When more than two target types are considered (i.e. n versus one combat), the theory of SVIC must be used to determine the order in which it is optimal to annihilate target types. This was not essential for the simplest case of $n = 2$.

Secondly, the algebraic method (see Section 4 of Appendix A) that we have developed for the synthesis of optimal policies appears to us as the only way to treat such problems. It should suffice for us to point out that Isbell and Marlow [6] (see also [7]) failed to discover the dispersal surfaces in the fire programming problem's solution when they used their geometric approach. Additionally, we do not feel that this geometric approach can be readily extended to problems with n target types ($n > 2$), whereas our algebraic approach apparently can.

Our re-examination of the Isbell-Marlow problem has been particularly fruitful, since it revealed the dominated payoff for $0 \leq \delta < R - \sqrt{R(R-1)}$, i.e. $P^o \in D(C_5) \cap D(C_5^S)$ implies that $P_5 > P_5^S$.

We had previously observed such a phenomenon in the supporting weapon system game of H. K. Weiss [11] (see [10] and also Appendix B of [8]). At that time, we could not relate such a dominated payoff in a differential game to any corresponding phenomenon in the simpler one-sided (optimal control) case. (It should be noted that Weiss did not use the subsequently well-known "in the small" necessary conditions of optimality from generalized control theory [4] (i.e. optimal control/ differential game theory) to develop his solution in [11] (it was erroneously reported that he did by S. Sternberg in [7]).)

Finally, we would like to point out that our work here lays a firm foundation for considering two-sided problems. We propose this to ONR as a future research task. It should be noted that Sternberg's work [7] does not cite any of the essential theory of SVIC. In Appendix E we have pointed out that lack of consideration of the theory of SVIC is a gap in the current theory of differential games (particularly as applied to tactical allocation problems in the Lanchester theory of combat).

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Appendix G. The Prescribed Duration Battle.

1. Introduction.

One of the objectives of this research sponsored by ONR is to investigate the sensitivity of the optimal target selection policy to the form of the combat attrition model. Of particular interest is whether for the simplest target selection problem the structure of the optimal policy is sensitive to whether combat attrition is modelled as a deterministic or a stochastic process. In Appendix I we consider the optimal control of the Lanchester stochastic process for a prescribed duration battle between a homogeneous Y-force and a heterogeneous enemy, X_1 and X_2. In order to compare the structures of the optimal target selection policy for deterministic and stochastic attrition processes, it is, of course, necessary to develop a solution to the deterministic problem. We do that in this appendix.

We have previously considered (several versions of) the prescribed duration battle [3], [4] (reproduced in this report as Appendices B and C). Our subsequent examination here has led to a revision of one of our earlier tentative conclusions about the structure of the optimal target selection policy: we now conclude that when target types undergo a "square-law" attrition process (as we consider here) the optimal allocation may depend (indirectly) upon the force levels. We have reached this conclusion after a careful study of the problem at hand.

Also of interest as regards the sensitivity of the structure of the optimal policy is whether the nature of the scenario (i.e. planning horizon) affects the structure. In [3] we compare the optimal target selection policy for the prescribed duration battle with that for a terminal control battle (a battle which only ends when force levels have reached some specified terminal state). After our re-examination here, we again will do this.

In view of the extensive details on basic necessary conditions of optimality and synthesis of the optimal policy for similar problems available elsewhere in this report (see, in particular, Appendices A, E, and F), we will take the liberty of outlining some results here. As it is, our documentation of the problem's solution is fairly lengthy. We may possibly present complete details in a future report.

Finally, we note that a key aspect in our obtaining these new results has been our use of the theory of state variable inequality constraints (SVIC). The reader can find a fairly extensive discussion of this theory and its application to target selection problems in the Lanchester theory of combat in Appendix E.

2. The Optimal Control Problem.

We consider the problem

$$\text{maximize} \quad \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with} \quad T_1 \quad \text{specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2, \qquad (1)$$

$$x_1, x_2, y \geq 0, \qquad 0 \leq \phi \leq 1, \qquad \text{and} \quad T \leq T_1,$$

where all symbols are defined in the next section.

The battle lasts for $0 \leq t \leq T_1$ unless, of course, one side or the other is annihilated before T_1. To be more precise, the battle terminates under one of the following three conditions:

(1) $x_1(T) = x_2(T) = 0$ and $T \leq T_1$,

(2) $y(T) = 0$ and $T \leq T_1$,

(3) $T = T_1$,

where T denotes the time at which the battle ends. Upon further analysis, it has been convenient to consider that there are eight "terminal states," or "target sets." These are shown in Table I. The reader should note that for S_4 through S_8 the battle ends when the force levels reach given terminal conditions. For these

Table I. Definition of Terminal States for Prescribed Duration Battle.

S_1: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, $T = T_1$

S_2: $x_1(T) = x_1(t_1) = 0$, $x_2(T) > 0$, $y(T) > 0$, $T = T_1$
where $t_1 < T$

S_3: $x_1(T) = x_1(t_3) > 0$, $x_2(T) = 0$, $y(T) > 0$, $T = T_1$
where $t_3 < T$

S_4: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$, $T \leq T_1$

S_5: $x_1(T) = x_1(t_1) = 0$, $x_2(T) > 0$, $y(T) = 0$, $T \leq T_1$
where $t_1 < T$

S_6: $x_1(T) = x_1(t_2) > 0$, $x_2(T) = 0$, $y(T) = 0$, $T \leq T_1$
where $t_2 < T$

S_7: $x_1(T) = x_1(t_1) = 0$, $x_2(T) = 0$, $y(T) > 0$, $T \leq T_1$
where $t_1 < T$

S_8: $x_1(T) = 0$, $x_2(T) = x_2(t_4) = 0$, $y(T) > T_1$
where $t_4 < T$

terminal states, T is undetermined, since it is determined by entry to the terminal state, and this depends upon the control used. For these cases, the well-known transversality condition must hold.

3. Notation.

The symbols which are used in this appendix are defined as follows:

$A = A(R,z) = [z^2(R-1) - R]/(z-1)^2$,

$B = B(R,z) = A(z-1)^2/z^2 = [z^2(R-1) - R]/z^2$,

a_1, a_2, b_1, b_2 = constant attrition-rate coefficients,

$D(S_i)$ = domain of controllability for S_i,

p, q, r = utilities assigned to surviving X_1, X_2 and Y forces respectively,

$p_i(t)$ for $i = 1,2,3$ = dual variable corresponding to $x_i(t)$ ($x_3(t) = y(t)$),

P_i for $i = 1, \ldots, 8$ = payoff associated with extremal leading to S_i,

$P^o = (x_1^o, x_2^o, y_0)$ = point in the initial state space,

$R = a_1 b_1/(a_2 b_2)$,

S_i for $i = 1, \ldots, 8$ = the $i^{\underline{th}}$ part of the terminal surface as defined in Table I,

$s = s(x_1^o, x_2^o) = b_1 x_1^o + b_2 x_2^o$,

t_1 = time at which X_1 is annihilated, i.e. $x_1(t_1) = 0$,

t_2 = first time at which $2b_1 x_1(t_2) x_2^o + b_2 (x_2^o)^2 = a_2 y^2(t_2)$ for an extremal leading to S_6,

t_3 = last time at which fire is directed at X_1 for an extremal leading to S_3,

t_4 = time at which X_2 is annihilated (before X_1), i.e. $x_2(t_4) = 0$, for an extremal leading to S_8,

T = time at which battle ends,

T_1 = Maximum possible duration for battle, i.e. $T \leq T_1$,

T_0 = lower bound on T_1 (developed from condition $x_2(T) = 0$) for an extremal leading to S_3,

$v = v(\tau) = a_2 p_2(\tau) - a_1 p_1(\tau)$,

x_1, x_2, y = average force strengths; with initial values x_1^o, x_2^o, y_0,

$z = \cosh \sqrt{a_2 b_2}\ \tau_1 (S_4) = \frac{R-\delta}{R-1}$,

$\alpha = \frac{r}{q} \sqrt{\frac{b_2}{a_2}}$,

$\delta = a_1 p/(a_2 q)$,

ϕ = fraction of Y-fire directed at X_1,

τ = "backwards time" from the end of the battle defined by $\tau = T-t$, i.e. the time remaining before the end of the battle,

$\tau_1(S_i)$ = "backwards time" of the first switch in tactics for extremals leading to S_i,

Additionally, remarks similar to those for $\tau_1(S_i)$ above apply to $t_1(S_i)$, $T(S_i)$, etc.

4. Outline of Solution Procedure.

Before giving our solution algoritm which is based upon Pontryagin's maximum principle* [2], it seems appropriate to define some terms. We have then

Definition 1: By an extremal path we mean a path on which the necessary conditions of optimality are everywhere satisfied (we use the word everywhere, since we assume that the class of admissible control functions is the class of piecewise-continuous functions).

Definition 2: By extremal control we mean the control used in order that the system follow an extremal path.

Definition 3: By the domain of controllability of a given terminal state we mean that subset of the initial state space from which extremals lead to the terminal state.

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*We use the form commonly used in this country. There is a sign difference between developments in the United States and those in the Soviet Union (see p. 108 of [1]).

Definition 4: By the <u>synthesis of the optimal control</u>, we mean the explicit determination of the time history of the optimal control from initial to terminal time (possibly as a function of initial conditions).

Our solution algorithm then is as follows:

(a) extremal control is determined by maximizing the Hamiltonian; since the state variables (force strengths) are non-negative, the control depends, in many cases, only on relationships between the dual variables (marginal return from destroying targets),

(b) from each separate terminal state, the time history of the dual variables is obtained by a backward integration of the adjoint system of differential equations combined with the extremal control; for a square law attrition process, the adjoint equations are independent of the state variables; the corner conditions must be considered in this step,

(c) for each terminal state the domain of controllability is determined by forward integration of the state equations using the time history of extremal control developed in (b); changes in control with time (existence of transition surface) may have to be considered in this step.

(d) the solution is determined at this point for regions of the initial state space which are covered by part of the domain of controllability for only one terminal state; one must also verify that the entire initial state space has been accounted for, since otherwise one may have overlooked some type of "singular" surface or even a terminal state,

(e) if domains of controllability overlap so that for a point of the initial state space contained in their intersection there is more than one extremal leading to the terminal surface, then one computes the payoff associated with each extremal; the optimal trajectory is selected from the extremals by comparing these values.

The above solution algorithm is essentially the same as that presented in Section 4 of Appendix A. Let us make a few remarks about the application of this procedure to the prescribed duration battle. In this case, we may think of time as being an additional

state variable. On the other hand, in the Isbell-Marlow terminal control problem, time may be considered to be a parameter and consequently was eliminated from the determinations in step (c) above. In other words, in the Isbell-Marlow problem a domain of controllability was determined by inequalities involving the three state variables; in the prescribed duration battle such a determination as in step (c) above involves the four variables T, x_1^o, x_2^o, and y_0. For the prescribed duration battle we have not obtained (relatively) simple analytic expressions in step (c) of the above procedure. Consequently, we could not use analytic methods to accomplish steps (d) and (e). In other words, we have not been able to analytically determine when the domains of controllability corresponding to different terminal states "overlap." Hence, in many cases we have analytically only been able to determine in extremal control.

However, we can use computational methods to determine the optimal control. We have expressed our "solution" to this problem (presented in the next section) so that we can readily to the following: given a point $P^o = (x_1^o, x_2^o, y_0)$ in the initial state space and T_1 we can determine which terminal states are reached by extremals. In other words, we can determine to which terminal states are reached by extremals. In other words, we can determine to which domains of controllability P^o belongs. Then, using the extremal control, we can compute numerically the payoff associated with each extremal and select the optimal policy from a finite number of possibilities. A M.S. thesis student (Robert L. Powers,

Lt., USN) is developing a digital computer program to use the "solution" of Tables II and III to determine the optimal target selection policy. Using this computer program, we plan to examine how much difference there is between payoffs from using optimal and non-optimal policies.

5. Summary of Solution.

We have applied our solution procedure of Section 4 to develop a "solution" in the sense discussed there. Without loss of generality, we may assume that $a_1b_1 > a_2b_2$, i.e. $R > 1$. Then, there are two cases to be considered

$$(1) \quad \delta \geq 1,$$

$$\text{and} \quad (2) \quad 0 \leq \delta < 1,$$

where $\delta = a_1p/(a_2q)$.

For Case (1): $\delta \geq 1$, the solution is shown in Tables II. In these tables, the domain of controllability and optimal policy is given for each terminal state. Since the domains of controllability don't overlap, the extremals are unique. Since it may be shown that a solution exists for this optimization problem, the unique extremal control is consequently the optimal control. Moreover, the optimal policy may be expressed in a particularly simple form: always concentrate all fire on X_1 while $x_1 > 0$. We have presented complete details on the domain of controllability for each terminal state and given all "event" time for two reasons: (1) to be

Table II. Solution to Prescribed Duration Battle for $\delta \geq 1$.

Nonrestrictive assumption: $R > 1$, i.e. $a_1b_1 > a_2b_2$

Case (1): $\delta \geq 1$ where $\delta = a_1p/(a_2q)$

S_1: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, $T = T_1$

Optimal Control: $\phi^*(t) = 1$ for $0 \leq t \leq T$

Case 1. $a_1b_1y_0^{\,2} > s^2$

$$T_1 < \frac{1}{\sqrt{a_1b_1}} \ln \left\{ \frac{\sqrt{a_1b_1y_0^{\,2} - s^2 + \left(b_2x_2^o\right)^2} - b_2x_2^o}{\sqrt{a_1b_1}\, y_0 - s} \right\}$$

Case 2. $a_1b_1y_0^{\,2} < s^2$

Subcase A. $a_1b_1y_0^{\,2} \geq s^2 - (b_2x_2^o)^2$

$$T_1 < \frac{1}{\sqrt{a_1b_1}} \ln \left\{ \frac{b_2x_2^o - \sqrt{a_1b_1y_0^{\,2} - s^2 + \left(b_2x_2^o\right)^2}}{s - \sqrt{a_1b_1}\, y_0} \right\}$$

Subcase B. $a_1b_1y_0^{\,2} < s^2 - \left(b_2x_2^o\right)^2$

$$T_1 < \frac{1}{\sqrt{a_1b_1}} \tanh^{-1} \left\{ \frac{\sqrt{a_1b_1}\, y_0}{s} \right\}$$

Case 3. $a_1b_1y_0^{\,2} = s^2$

$$T_1 < \frac{1}{\sqrt{a_1b_1}} \ln \left\{ \frac{s}{b_2x_2^o} \right\}$$

Table II. Solution to Prescribed

Duration Battle for $\delta \geq 1$.
(cont.) - 1

$$S_2:\ x_1(T) = x_1(t_1) = 0,\ x_2(T) > 0,\ y(T) > 0,\quad T = T_1$$

where $t_1 < T$

$$\text{Optimal Control:}\quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \quad \text{where } x_1(t_1) = 0 \\ 0 & \text{for } t_1 < t \leq T \end{cases}$$

Always must have $a_1 b_1 y_0^{\ 2} > s^2 - (b_2 x_2^o)^2$

Case 1. $a_1 b_1 y_0^{\ 2} \geq s^2 + (R-1)(b_2 x_2^o)^2$

$$t_1 \leq T_1 < t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{b_2 x_2^o \sqrt{R}}{\sqrt{a_1 b_1 y_0^{\ 2} - s^2 + \left(b_2 x_2^o\right)^2}}\right\}$$

Case 2. $a_1 b_1 y_0^{\ 2} \leq s \ + (R-1)(b_2 x_2^o)^2$

$$t_1 \leq T_1 < t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{\sqrt{a_1 b_1 y_0^{\ 2} - s^2 + (b_2 x_2^o)^2}}{b_2 x_2^o \sqrt{R}}\right\}$$

where t_1 is given by

(1) for $a_1 b_1 y_0^{\ 2} > s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{\frac{\sqrt{a_1 b_1 y_0^{\ 2} - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\ y_0 - s}\right\}$$

(2) for $a_1 b_1 y_0^{\ 2} < s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{\frac{b_2 x_2^o - \sqrt{a_1 b_1 y_0^{\ 2} - s^2 + (b_2 x_2^o)^2}}{s - \sqrt{a_1 b_1}\ y_0}\right\}$$

Table II. Solution to Prescribed Duration Battle for $\delta \geq 1$. (cont.) - 2

S_2: (concluded)

(3) for $a_1 b_1 y_0{}^2 = s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ell n \left\{ \frac{s}{b_2 x_2^o} \right\}$$

S_3: $x_1(T) = x_1(t_3) > 0$, $x_2(T) = 0$, $y(T) > 0$, $T = T_1$

where $t_3 < T$

NO EXTREMALS LEAD TO THIS END STATE

S_4: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$, $T \leq T_1$

ptimal Control: $\phi^*(t) = 1$ for $0 \leq t \leq T$

omain of Controllability: $a_1 b_1 y_0{}^2 < s^2 - (b_2 x_2^o)^2$

$$T = \frac{1}{\sqrt{a_1 b_1}} \tanh^{-1} \left\{ \frac{\sqrt{a_1 b_1} \quad y_0}{s} \right\} \leq T_1$$

Table II. Solution to Prescribed

Duration Battle for $\delta \geq 1$.
(cont.) -3

S_5: $x_1(T) = x_1(t_1) = 0,\ x_2(T) > 0,\ y(T) = 0,\ T \leq T_1$

where $t_1 < T$

Optimal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \text{ where } x_1(t_1) = 0 \\ 0 & \text{for } t_1 < t \leq T \end{cases}$

Domain of Controllability: $a_1 b_1 y_0{}^2 \geq s^2 - (b_2 x_2^o)^2$

$a_1 b_1 y_0{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1} \left\{ \frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}{b_2 x_2{}^o \sqrt{R}} \right\} \leq T_1$$

where t_1 is given by

(1) for $a_1 b_1 y_0{}^2 > s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s} \right\}$$

(2) for $a_1 b_1 y_0{}^2 < s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{b_2 x_2^o - \sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}{s - \sqrt{a_1 b_1}\, y_0} \right\}$$

(3) for $a_1 b_1 y_0{}^2 = s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{s}{b_2 x_2^o} \right\}$$

Table II. Solution to Prescribed Duration Battle for $\delta \geq 1$. (concluded) -4

S_6: $x_1(T) = x_1(t_2) > 0,\ x_2(T) = 0,\ y(T) = 0,\ T \leq T_1$

where $t_2 < T$

NO EXTREMALS LEAD TO THIS END STATE

S_7: $x_1(T) = x_1(t_1) = 0,\ x_2(T) = 0,\ y(T) > 0,\ T \leq T_1$

where $t_1 < T$

$$\text{Optimal Control:}\quad \phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \quad \text{where } x_1(t_1) = 0 \\ 0 & \text{for } t_1 < t \leq T \end{cases}$$

Domain of Controllability: $a_1 b_1 y_0{}^2 > s^2 + (R-1)(b_2 x_2^o)^2$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{ \frac{b_2 x_2^o \sqrt{R}}{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}} \right\} \leq T_1$$

where t_1 is given by

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\ y_0 - s} \right\}$$

S_8: $x_1(T) = 0,\ x_2(T) = x_2(t_4) = 0,\ y(T) > 0,\ T \leq T_1$

where $t_4 < T$

NO EXTREMALS LEAD TO THIS END STATE

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able to make numerical comparisons with the stochastic control problem considered in Appendix I, and (2) to be able to perform a sensitivity analysis on target selection policies.

For Case (2): $0 \leq \delta < 1$, the solution is shown in Tables III. It is expressed in the same format as for Case (1) with the exception that the controls are extremal (with two exceptions noted below).

It should be recalled that for the Isbell-Marlow terminal control problem (see Appendix F) when $0 \leq \delta < R - \sqrt{R(R-1)}$ some domains of controllability overlapped. Hence, extremals were not unique and considerations "in the large" had to be used to determine the optimal policy. We suspect that similar behavior happens in the prescribed duration battle, since we may consider the Isbell-Marlow problem to be imbedded in the prescribed duration battle (i.e. when $T < T_1$, the solution to the Isbell-Marlow problem is applicable). Moreover, lack of explicit analytic expressions for two domains of controllability (that for S_1 when there is a switch in tactics and S_3) has prevented explicit determination as to which overlap. Furthermore, the computation of the payoff associated with each (overlapping) extremal has not been analytically tractable. Hence, (except for two instances) extremal controls and (either implicit or explicit) expressions for the corresponding domains of controllability are given in Tables III. However, as discussed in Section 4 above, one can use the information presented in Tables III to numerically determine an optimal target selection policy for any specific set of input values.

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$.

Nonrestrictive assumption: $R > 1$, i.e. $a_1 b_1 > a_2 b_2$

Case (2): $0 \le \delta < 1$ where $\delta = a_1 p/(a_2 q)$

S_1: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, $T = T_1$

Since $y(T) > 0$, the switching time $\tau_1 = \tau_1(S_1)$ is given by

$$\tau_1 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right\},$$

where $z = \dfrac{R - \delta}{R - 1}$ and $\alpha = \dfrac{r}{q}\sqrt{\dfrac{b_2}{a_2}}$

Case 1. $\tau_1 \ge T$

Extremal Control: $\phi^*(t) = 0$ for $0 \le t \le T$

Subcase A. $a_1 b_1 y_0{}^2 < R\left\{ s^2 - (b_1 x_1^o)^2 \right\}$

$$T_1 < \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1} \left\{ \frac{\sqrt{a_1 b_1}\; y_0}{s\sqrt{R}} \right\}$$

Subcase B. $a_1 b_1 y_0{}^2 \ge R\left\{ s^2 - (b_1 x_1^o)^2 \right\}$

(1) for $a_1 b_1 y_0{}^2 > Rs^2$

$$T_1 < \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0{}^2 - R\{s^2 - (b_1 x_1^o)^2\}} - b_1 x_1{}^o \sqrt{R}}{\sqrt{a_1 b_1}\; y_0 - s\sqrt{R}} \right\}$$

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)-1

S_1: (continued)

(2) for $a_1 b_1 y_0{}^2 < Rs^2$

$$T_1 < \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{b_1 x_1^o \sqrt{R} - \sqrt{a_1 b_1 y_0{}^2 - R\{s^2 - (b_1 x_1^o)^2\}}}{s\sqrt{R} - \sqrt{a_1 b_1}\; y_0} \right\}$$

(3) for $a_1 b_1 y_0{}^2 = Rs^2$

$$T_1 < \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{s}{b_2 x_2^o} \right\}$$

Case 2. $\tau_1 < T$

Extremal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T - \tau_1 \\ 0 & \text{for } T - \tau_1 < t \le T \end{cases}$

Require that $b_1 x_1(T - \tau_1) > 0$, $b_2 x_2(T) > 0$, $y(T) > 0$.

where

$$b_1 x_1(T-\tau) = s \cosh\sqrt{a_1 b_1}(T-\tau_1) - \sqrt{a_1 b_1}\; y_0 \sinh\sqrt{a_1 b_1}(T-\tau_1) - b_2 x_2^o,$$

$$b_2 x_2(T) = \{s \cosh\sqrt{a_1 b_1}(T-\tau_1) - \sqrt{a_1 b_1} y_0 \sinh\sqrt{a_1 b_1}(T-\tau_1)\}(\cosh\sqrt{a_2 b_2}\;\tau_1 - 1)$$

$$-\sqrt{a_2 b_2}\{y_0 \cosh\sqrt{a_1 b_1}(T-\tau_1) - \frac{s}{\sqrt{a_1 b_1}} \sinh\sqrt{a_1 b_1}(T-\tau_1)\}\sinh\sqrt{a_2 b_2}\;\tau_1 + b_2 x_2^o,$$

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)-2

S_1: (concluded)

$$y(T) = \left\{ y_0 \cosh\ a_1b_1(T-\tau_1) - \frac{s}{\sqrt{a_1b_1}} \sinh \sqrt{a_1b_1}(T-\tau_1) \right\} \cosh \sqrt{a_2b_2}\ \tau_1$$

$$-\left\{ s \cosh \sqrt{a_1b_1}\ (T-\tau_1) - \sqrt{a_1b_1}\ y_0 \sinh \sqrt{a_1b_1}(T-\tau_1) \right\} \frac{\sinh\sqrt{a_1b_2}\ \tau_1}{\sqrt{a_2b_2}}$$

S_2: $x_1(T) = x_1(t_1) = 0, \quad x_2(T) > 0,\ y(T) > 0, \quad T = T_1$

where $t_1 < T$

Extremal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \quad \text{where } x_1(t_1) = 0 \\ 0 & \text{for } t_1 < t \le T \end{cases}$

Always must have $a_1b_1{y_0}^2 > s^2 - (b_2x_2^o)^2$

$$t_1 + \tau_1 \le T_1$$

Case 1. $a_1b_1{y_0}^2 \ge s^2 + (R-1)(b_2x_2^o)^2$

$$T_1 < t_1 + \frac{1}{\sqrt{a_2b_2}} \tanh^{-1} \left\{ \frac{b_2{x_2}^o\sqrt{R}}{\sqrt{a_1b_1{y_0}^2 - s^2 + (b_2x_2^o)^2}} \right\}$$

Case 2. $a_1b_1{y_0}^2 < s^2 + (R-1)(b_2x_2^o)^2$

$$T_1 < t_1 + \frac{1}{\sqrt{a_2b_2}} \tanh^{-1} \left\{ \frac{\sqrt{a_1b_1{y_0}^2 - s^2 + (b_2x_2^o)^2}}{b_2x_2^o\ \sqrt{R}} \right\}$$

where $\tau_1 = \tau_1(S_1)$ is given by

$$\tau_1 = \frac{1}{\sqrt{a_2b_2}} \ell n \left\{ \frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right\},$$

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)-3

S_2: (concluded)

and $z = \frac{R-\delta}{R-1}$, $\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}}$,

and where $t_1 = t_1(S_2)$ is given by

(1) for $a_1 b_1 y_0{}^2 > s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s} \right\}$$

(2) for $a_1 b_1 y_0{}^2 < s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{b_2 x_2^o - \sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}{s - \sqrt{a_1 b_1}\, y_0} \right\}$$

(3) for $a_1 b_1 y_0{}^2 = s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{s}{b_2 x_2^o} \right\}$$

S_3: $x_1(T) = x_1(t_3) > 0$, $x_2(T) = 0$, $y(T) > 0$, $T = T_1$

where $t_3 < T$

Extremal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_3 \\ 0 & \text{for } t_3 < t \le T \end{cases}$ where explicit expression for t_3 is given below.

Always must have $a_1 b_1 y_0^2 \ge R\,\{s^2 - (b_1 x_1^o)^2\}$

$t_3 \ge T_1 - \tau_1$ (see below)

Case 1. $a_1 b_1 y_0{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$

$T_0 \le T_1 < T(S_6)$

Table III. Solution to Prescribed

Duration Battle for $0 \le \delta < 1$
(cont.)-4

S_3: (continued)

where T_0 is given below

$$T(S_6) = t_2(S_6) + \frac{1}{\sqrt{a_2 b_2}} \cosh^{-1}\left\{\frac{Rb_2x_2^o + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}{(R-1)b_2x_2^o + \sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}}\right\}$$

$$y^2(t_2) = y^2\,(t_2(S_6)) = \frac{x_2^o}{a_2}\left\{(2R-1)\; b_2x_2^o \;+ 2\sqrt{s^2 + R(R-1)(b_2x_2^o)^2 - a_1b_1y_0^2}\right\}$$

and where $t_2 = t_2(S_6)$ is given by

(1) for $a_1b_1y_0^2 > s^2$

$$t_2(S_6) = \frac{1}{\sqrt{a_1b_1}}\,\ell n\left\{\frac{y(t_2) - \sqrt{y^2(t_2) - y_0^2 + s^2/(a_1b_1)}}{y_0 - s/\sqrt{a_1b_1}}\right\}$$

(2) for $a_1b_1y_0^2 < s^2$

$$t_2(S_6) = \frac{1}{\sqrt{a_1b_1}}\,\ell n\left\{\frac{\sqrt{y^2(t_2) - y_0^2 + s^2/(a_1b_1)} - y(t_2)}{s/\sqrt{a_1b_1} - y_0}\right\}$$

(3) for $a_1b_1y_0^2 = s^2$

$$t_2(S_6) = \frac{1}{\sqrt{a_1b_1}}\,\ell n\left\{\frac{y_0}{y(t_2)}\right\}$$

Case 2. $a_1b_1y_0^2 \ge s^2 + (R-1)(b_2x_2^o)^2$

$T_0 \le T_1 < T\,(S_7)$

where T_0 is given below

$$T(S_7) = t_1(S_7) + \frac{1}{\sqrt{a_2b_2}}\,\tanh^{-1}\left\{\frac{b_2x_2^o\,\sqrt{R}}{\sqrt{a_1b_1y_0^2 - s^2 + (b_2x_2^o)^2}}\right\}$$

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)-5

S_3: (continued)

and where $t_1 = t_1(S_7)$ is given by

$$t_1(S_7) = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\; y_0 - s} \right\}$$

In both the above Cases 1. and 2. we have T_0 given by

(1) for $a_1 b_1 y_0^2 > Rs^2$

$$T_0 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{\sqrt{a_1 b_1 y_0^2 - R\{s^2 - (b_1 x_1^o)^2\}} - b_1 x_1^o \sqrt{R}}{\sqrt{a_1 b_1}\; y_0 - s\sqrt{R}} \right\}$$

(2) for $a_1 b_1 y_0^2 < Rs^2$

$$T_0 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{b_1 x_1^o \sqrt{R} - \sqrt{a_1 b_1 y_0^2 - R\{s^2 - (b_1 x_1^o)^2\}}}{s\sqrt{R} - \sqrt{a_1 b_1}\; y_0} \right\}$$

(3) for $a_1 b_1 y_0^2 = Rs^2$

$$T_0 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{s}{b_2 x_2^o} \right\}$$

Furthermore, we must have

$$t_3 \ge T_1 - \tau_1 \qquad \text{(i.e. } T_1 - t_3 \le \tau_1)$$

where $\tau_1 = \tau_1(S_1)$ is given by

$$\tau_1 = \frac{1}{\sqrt{a_2 b_2}} \ln \left\{ \frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right\},$$

and $z = \dfrac{R-\delta}{R-1}$, $\alpha = \dfrac{r}{q}\sqrt{\dfrac{b_2}{a_2}}$,

and where $t_3 = t_3(S_3)$ is a root of the equation

$$F(t_3) = 0,$$

Table III. Solution to Prescribed

Duration Battle for $0 \leq \delta < 1$. (cont.)- 6

S_3: (continued)

and we have

$$F(t_3) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{R}}\right)\Big[s \cosh\Big\{\sqrt{a_2b_2}\,(T_1 - t_3) + \sqrt{a_1b_1}\,t_3\Big\}$$

$$- \sqrt{a_1b_1}\,y_0 \sinh\Big\{\sqrt{a_2b_2}\,(T_1 - t_3) + \sqrt{a_1b_1}\,t_3\Big\}\Big]$$

$$+ \frac{1}{2}\left(1 - \frac{1}{\sqrt{R}}\right)\Big[s \cosh\Big\{\sqrt{a_2b_2}\,(T_1 - t_3) - \sqrt{a_1b_1}\,t_3\Big\}$$

$$+ \sqrt{a_1b_1}\,y_0 \sinh\Big\{\sqrt{a_2b_2}\,(T_1 - t_3) - \sqrt{a_1b_1}\,t_3\Big\}\Big]$$

$$- s \cosh \sqrt{a_1b_1}\,t_3 + \sqrt{a_1b_1}\,y_0 \sinh \sqrt{a_1b_1}\,t_3 + b_2x_2^0.$$

Solve the above equation $F(t_3)=0$ by Newton-Raphson method.

$$t_3^{(n+1)} = t_3^{(n)} - \frac{F\left(t_3^{(n)}\right)}{F'\left(t_3^{(n)}\right)},$$

where

$$F'(t_3) = \frac{\sqrt{a_1b_1}}{2}\left(1 - \frac{1}{R}\right)\Big[s \sinh\{\sqrt{a_2b_2}\,(T_1 - t_3) + \sqrt{a_1b_1}\,t_3\}$$

$$- \sqrt{a_1b_1}\,y_0 \cosh\{\sqrt{a_2b_2}\,(T_1 - t_3) + \sqrt{a_1b_1}\,t_3\}\Big]$$

$$- \frac{\sqrt{a_1b_1}}{2}\left(1 - \frac{1}{R}\right)\Big[s \sinh\{\sqrt{a_2b_2}\,(T_1 - t_3) - \sqrt{a_1b_1}\,t_3\}$$

$$+ \sqrt{a_1b_1}\,y_0 \cosh\{\sqrt{a_2b_2}\,(T_1 - t_3) - \sqrt{a_1b_1}\,t_3\}\Big]$$

$$- s\sqrt{a_1b_1} \sinh \sqrt{a_1b_1}\,t_3 + a_1b_1y_0 \cosh \sqrt{a_1b_1}\,t_3.$$

Table III. Solution to Prescribed Duration Battle for $0 \leq \delta < 1$. (cont.) - 7

S_3: (concluded)

There are two cases to consider for starting value for Newton-Raphson method, $t_2^{(o)}$:

Case a. $a_1 b_1 y_0^2 < s^2 + (R - 1)(b_2 x_2^o)^2$

$$t_3^{(o)} = t_2(S_6) - \left\{T(S_6) - T_1\right\},$$

where $t_2(S_6)$ and $T(S_6)$ are given above.

Case b. $a_1 b_1 y_0^2 \geq s^2 + (R - 1)(b_2 x_2^o)^2$

$$t_3^{(o)} = t_1(S_7) - \left\{T(S_7) - T_1\right\},$$

where $t_1(S_7)$ and $T(S_7)$ and given above.

NOTE: No extremal leads to S_3 if:

(1) there is no root to the equation $F(t_3) = 0$,

or

(2) we do not have $0 \leq t_3 \leq T_1$,

or

(3) we do not have

$$x_1(t_3) > 0,$$

$$x_2(T_1) = 0,$$

$$y(T_1) > 0.$$

Table III. Solution to Prescribed Duration Battle for $0 \leq \delta < 1$. (cont.) - 8

S_4: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$, $T \leq T_1$

Since $y(T) = 0$, the switching time $\tau_1 = \tau_1(S_4)$ is given by

$$\tau_1 = \tau_1(S_4) = \frac{1}{\sqrt{a_2 b_2}} \cosh^{-1}\left(\frac{R-\delta}{R-1}\right) = \frac{1}{\sqrt{a_2 b_2}} \ln\left(z + \sqrt{z^2 - 1}\right),$$

$$\text{where } z = \frac{R-\delta}{R-1}.$$

Case 1. $\tau_1 \geq T$

Extremal Control: $\phi^*(t) = 0$ for $0 \leq t \leq T$,

Domain of Controllability: $a_1 b_1 y_0^2 \leq Rs^2\left\{1 - 1/z^2\right\}$,

$$a_1 b_1 y_0^2 < R\left\{s^2 - (b_1 x_1^o)^2\right\},$$

$$a_1 b_1 y_0^2 < s^2 + (R-1)(b_2 x_2^o)^2;$$

$$T = \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{\sqrt{a_2 b_2}\, y_0}{s}\right\} \leq T_1.$$

Note: for $0 \leq \delta < R - \sqrt{R(R-1)}$ optimal pathes also satisfy (equality yielding the <u>dispersal surface</u>) for $0 \leq x_1^o < b_2 x_2^o / (k\, b_1)$

$$a_1 b_1 y_0^2 \leq Rs^2 - \frac{R}{z^2}\left\{b_1 x_1^o \frac{[z^2(R-1) + R]}{2R} + b_2 x_2^o\right\}^2,$$

where k is given by $k = \left\{z^2 - R(z-1)^2\right\} / (2R)$.

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.) - 9

S_4: (continued)

Case 2. $\tau_1 < T$

Subcase A. $R-\sqrt{R(R-1)} < \delta < 1$

Optimal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le T-\tau_1 \\ 0 & \text{for } T-\tau_1 < \tau \le T \end{cases}$ where $\tau_1 = \tau_1(S_4)$ is given above

Domain of Controllability: $a_1 b_1 y_0{}^2 > s^2 + A(b_2 x_2^o)^2$

$$a_1 b_1 y_0{}^2 < s^2 + B(b_2 x_2^o)^2$$

$$a_1 b_1 y_0{}^2 > Rs^2 \left\{1 - 1/z^2\right\}$$

$$T = \tau_1 + \frac{1}{2\sqrt{a_1 b_1}} \ln \left\{ \frac{(z - \sqrt{R(z^2-1)})(s + \sqrt{a_1 b_1}\, y_0)}{(s - \sqrt{a_1 b_1}\, y_0)(z + \sqrt{R(z^2-1)})} \right\} \le T_1$$

Subcase B. $\delta = R - \sqrt{R(R-1)}$

Optimal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_5 < t_1 = t_1(S_2) \\ 0 & \text{for } t_5 < t \le T \end{cases}$

where $t_1 = t_1(S_2) = \dfrac{1}{\sqrt{a_1 b_1}} \ln \left\{ \dfrac{s}{b_2 x_2^o} \right\}$

and t_5 is <u>any</u> value such that $0 \le t_5 < t_1$.

Domain of Controllability: $a_1 b_1\, y_0{}^2 = s^2$

$$b_2 x_2^o > b_1 x_1^o \left(\sqrt{\frac{R}{R-1}} - 1 \right)$$

Table III. Solution to Prescribed Duration Battle for $0 \leq \delta < 1$. (cont.)- 10

S_4: (concluded)

Note: In this case the optimal control is not unique. Consequently T is not unique, but it lies within the bounds

$$t_1(S_2) + \tau_1 < T \leq \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{\sqrt{a_2 b_2}\, y_0}{s}\right\}.$$

Furthermore, we must have $T \leq T_1$.

S_5: $x_1(T) = x_1(t_1) = 0$, $x_2(T) > 0$, $y(T) = 0$, $T \leq T_1$

Extremal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq t_1 \quad \text{where } x_1(t_1) = 0 \\ 0 & \text{for } t_1 < t \leq T \end{cases}$

Domain of Controllability: $a_1 b_1 y_0{}^2 \geq s^2 + B(b_2 x_2^o)^2$

$$a_1 b_1 y_0{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}{b_2 x_2^o \sqrt{R}}\right\} \leq T_1$$

where $t_1 = t_1(S_5) = t_1(S_2)$ is given by

(1) for $a_1 b_1 y_0{}^2 > s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{\frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s}\right\}$$

Table III. Solution to Prescribed

Duration Battle for $0 \le \delta < 1$.
(cont.)-11

S_5: (concluded)

(2) for $a_1 b_1 y_0{}^2 < s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{b_2 x_2^o - \sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}{s - \sqrt{a_1 b_1}\, y_0} \right\}$$

(3) for $a_1 b_1 y_0{}^2 = s^2$

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{s}{b_2 x_2^o} \right\}$$

NOTE: for $0 \le \delta < R - \sqrt{R(R-1)}$ optimal paths also satisfy (equality yielding the <u>dispersal surface</u>) for $0 \le x_2^o \le kb_1 x_1^o / b_2$

$$a_1 b_1 y_0{}^2 \ge Rs^2 - \frac{R}{z^2} \left\{ b_1 x_1^o \frac{[z^2(R-1) + R]}{2R} + b_2 x_2^o \right\}^2,$$

where k is given by $k = \left\{ z^2 - R(z-1)^2 \right\} / (2R)$.

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)- 12

S_6: $x_1(T) = x_1(t_2) > 0$, $x_2(T) = 0$, $y(T) = 0$, $T \le T_1$

where $t_2 < T$

Extremal Control: $$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_2 \\ 0 & \text{for } t_2 < t \le T \end{cases}$$

where $t_2 = t_2(S_6)$ is the smallest t such that

$$a_1 b_1 y^2(t_2) = R\left\{\left(b_1 x_1(t_2) + b_2 x_2^o\right)^2 - \left(b_1 x_1(t_2)\right)^2\right\}.$$

An explicit expression is given for t_2 below.

Domain of Controllability: $$a_1 b_1 y_0{}^2 \le s^2 + A(b_2 x_2^o)^2$$

$$a_1 b_1 y_0{}^2 \ge R\left\{s^2 - (b_1 x_1^o)^2\right\}$$

$$a_1 b_1 y_0{}^2 < s^2 + (R-1)(b_2 x_2^o)^2$$

$$T = t_2 + \frac{1}{\sqrt{a_2 b_2}} \cosh^{-1}\left\{\frac{R b_2 x_2^o + \sqrt{s^2 + R(R-1)(b_2 x_2^o)^2 - a_1 b_1 y_0{}^2}}{(R-1) b_2 x_2^o + \sqrt{s^2 + R(R-1)(b_2 x_2^o)^2 - a_1 b_1 y_0{}^2}}\right\} \le T_1$$

where $t_2 = t_2(S_6)$ is given below and $y(t_2)$ used in the computation of t_2 is given by

$$y(t_2) = \sqrt{\frac{x_2^o}{a_2}\left\{(2R-1) b_2 x_2^o + 2\sqrt{s^2 + R(R-1)(b_2 x_2^o)^2 - a_1 b_1 y_0{}^2}\right\}}\ .$$

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (cont.)- 13

S_6: (concluded)

Also, $t_2 = t_2(S_6)$ is given by

(1) for $a_1 b_1 y_0^2 > s^2$

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{y(t_2) - \sqrt{y^2(t_2) - y_0^2 + s^2/(a_1 b_1)}}{y_0 - s/\sqrt{a_1 b_1}} \right\}$$

(2) for $a_1 b_1 y_0^2 < s^2$

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{\sqrt{y^2(t_2) - y_0^2 + s^2/(a_1 b_1)} - y(t_2)}{s/\sqrt{a_1 b_1} - y_0} \right\}$$

(3) for $a_1 b_1 y_0^2 = s^2$

$$t_2 = \frac{1}{\sqrt{a_1 b_1}} \ln \left\{ \frac{y_0}{y(t_2)} \right\}$$

NOTE 1: We also have that $b_2 x_2^o (R-1) \le b_1 x_1^o$.

NOTE 2: for $0 \le \delta < R - \sqrt{R(R-1)}$ optimal paths also satisfy (there is a dispersal surface (whose equation hasn't been explicitly determined) in the solution)

$$0 \le x_2^o \le k b_1 x_1^o / b_2,$$

where k is given by $k = \{z^2 - R(z-1)^2\}/(2R)$.

Table III. Solution to Prescribed Duration Battle for $0 \le \delta < 1$. (concluded)- 14

S_7: $x_1(T) = x_1(t_1) = 0$, $x_2(T) = 0$, $y(T) > 0$, $T \le T_1$

where $t_1 < T$

Optimal Control: $\phi^*(t) = \begin{cases} 1 & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } t_1 < t \le T \end{cases}$ where $x_1(t_1) = 0$

Domain of Controllability: $a_1 b_1 y_0{}^2 > s^2 + (R-1)(b_2 x_2^o)^2$

$$T = t_1 + \frac{1}{\sqrt{a_2 b_2}} \tanh^{-1}\left\{\frac{b_2 x_2^o \sqrt{R}}{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2}}\right\} \le T_1$$

where t_1 is given by

$$t_1 = \frac{1}{\sqrt{a_1 b_1}} \ln\left\{\frac{\sqrt{a_1 b_1 y_0{}^2 - s^2 + (b_2 x_2^o)^2} - b_2 x_2^o}{\sqrt{a_1 b_1}\, y_0 - s}\right\}$$

S_8: $x_1(T) = 0$, $x_2(T) = x_2(t_4) = 0$, $y(T) > 0$, $T \le T_1$

where $t_4 < T$

NO EXTREMALS LEAD TO THIS END STATE

For S_4 (which corresponds to C_5 and C_5^S of the Isbell-Marlow problem) the optimal control can be determined explicitly when when there is a switch in tactics for $R - \sqrt{R(R-1)} < \delta < 1$. This is because the solution for S_4 is essentially the same as for the corresponding part of the Isbell-Marlow problem. For the latter problem (see Appendix F) we know that optimal paths lead to C_5^S only for $R - \sqrt{R(R-1)} < \delta < 1$.

In Case (2) the optimal policy cannot be expressed in the very simple form as in the first case. When Y wins in time less than T_1 (S_7 for which the optimal policy is determined) the solution is precisely the same as when $\delta \geq 1$. However, for all other cases (i.e. terminal states S_1 through S_6) the extremal policy is to finish the prescribed duration battle by firing at X_2, regardless of whether or not X_1 has been annihilated. This differs from that when $\delta \geq 1$. Thus, we see that the initial force levels may (indirectly) affect the optimal target selection policy.

Finally, in considering Tables III the following may be useful:

(a) for $0 \leq \delta < R - \sqrt{R(R-1)}$,

$$\text{we have} \quad A > B > 0, \tag{2}$$

(b) for $\delta = R - \sqrt{R(R-1)}$,

$$\text{we have} \quad A = B = 0, \tag{3}$$

(c) for $R - \sqrt{R(R-1)} < \delta < 1$,

$$\text{we have} \quad A < B < 0. \tag{4}$$

The quantities A and B appear in inequalities defining various domains of controllability and are defined by (see also Section 3)

$$A = A(R,z) = \frac{R(z^2 - 1) - z^2}{(z - 1)^2}, \tag{5}$$

and

$$B = B(R,z) = \frac{R(z^2 - 1) - z^2}{z^2}. \tag{6}$$

6. Development of Solution.

We will now discuss the application of our solution procedure which we outlined in Section 4. Because many of the developments parallel those already given in detail for the Isbell-Marlow problem (see Appendices A and F), we will omit (or outline) them. Possibly we may provide more details in the future. Several novel (and new) aspects, however, are examined in detail below.

In applying our solution algorithm one must first consider the basic necessary conditions of optimality. However, since we have done this in detail in Section 4.b.(1) of Appendix E (only the boundry conditions of the dual variables differ between the prescribed duration battle and the terminal control case), we will only summarize the main results here with the interested reader being directed to Appendix E for details.

We showed that in order for it to be optimal to have $x_1(t) = 0$ for a finite interval of time we must have

$$a_1b_1 \geq a_2b_2. \tag{7}$$

At entry (when $t = t_1$) to a constrained subarc on which $x_1 = 0$, we have (from the corner conditions)

$$p_1(t_1-) = \frac{a_2}{a_1} p_2 (t_1-), \tag{8}$$

$$p_i(t_1-) = p_i(t_1+) \text{ for } i = 2,3, \tag{9}$$

where t_1- denotes a left-hand limit.

Additionally, we showed that when $x_1 > 0$ and $x_2 > 0$ the extremal control (determined by the maximum principle) is given by

$$\phi^*(t) = \begin{cases} 1 & \text{for } v(t) > 0, \\ 0 & \text{for } v(t) < 0, \end{cases} \tag{10}$$

where

$$v(t) = a_1(-p_1(t)) - a_2(-p_2(t)) = a_2p_2(t) - a_1p_1(t). \tag{11}$$

Furthermore, $v(t) \neq 0$ almost everywhere. Differentiation of (11) and combination with the adjoint equations yields

$$\frac{dv}{dt} = -p_3(t)\ (a_1b_1 - a_2b_2). \tag{12}$$

Since it is readily shown that $p_3(t) > 0$ for $t < T$, we have that

$$\frac{dv}{dt} < 0 \text{ for all } t < T, \tag{13}$$

where our nonrestrictive assumption $a_1b_1 > a_2b_2$ should be recalled.

Considering the above, the time history of the extremal control is readily developed for each terminal state once the boundry conditions on the dual variables have been determined. For S_1, S_2, and S_3 the length of the battle is equal to T_1, and the boundry conditions of the dual variables are shown in Table IV. It is to be noted that

for S_1, S_2, and S_3

$$p_3 \ (t = T) = r > 0 \tag{14}$$

For S_4, S_5, and S_6 the length of the battle is determined by the time to annihilate Y so that $y(t = T) = 0$ is an equality constraint on a state variable at an unspecified terminal time. The boundry conditions of the dual variables are shown in Table V. It is to be noted that

for S_4, S_5, and S_6

$$p_3(t = T) = 0. \tag{15}$$

Moreover, for S_3, and S_6 we have that

$$\frac{a_1 p}{a_2} < -p_2(t = T) \leq q. \tag{16}$$

The synthesis of extremal control and determination of the domains of controllability are similar to developments in Appendices A and F. Consequently they are omitted here. There are two interesting aspects that we encountered in doing this work

Table IV. Boundry Conditions of Dual Variables for Terminal States S_1, S_2, and S_3.

For S_1, S_2, and S_3:

$$p_1(t = T) = -p + \nu_1,$$

where

$$\nu_1 \begin{cases} = 0 & \text{for } x_1(T) > 0, \\ \geq 0 & \text{for } x_1(T) = 0, \end{cases}$$

$$p_2(t = T) = -q + \nu_2,$$

where

$$\nu_2 \begin{cases} = 0 & \text{for } x_1(T) > 0, \\ \geq 0 & \text{for } x_1(T) = 0, \end{cases}$$

$$p_3(t = T) = r.$$

Table V. Boundry Conditions of Dual Variables for Terminal States S_4, S_5, and S_6.

For S_4, S_5, and S_6:

$$p_1(t = T) = -p + \nu_1,$$

where

$$\nu_1 \begin{cases} = 0 & \text{for } x_1(T) > 0, \\ \geq 0 & \text{for } x_1(T) = 0, \end{cases}$$

$$p_2(t = T) = -q + \nu_2,$$

where

$$\nu_2 \begin{cases} = 0 & \text{for } x_2(T) > 0, \\ \geq 0 & \text{for } x_2(T) = 0, \end{cases}$$

$p_3(t = T) = \nu_3$, where ν_3 is unrestricted.

However, the transversality condition

$$H(t = T, x_i, p_i, \phi^*(t = T)) = 0$$

yields that

$$p_3(t = T) = 0.$$

(most of the details of which we have omitted). These are

(a) when a switch in the target type upon which all Y-fire is concentrated occurs without the annihilation of a target type, the switching time depends upon the initial force levels and possibly the valuation of Y survivors.

(b) when $P^o = (x_1^o, x_2^o, y_0)$ is such that when $\delta < 1$ an extremal leads to S_4^S (i.e. we reach S_4 with a switch in tactics) with $T(S_4^S) < T_1$, we can also possibly steer the system to an end point with $y(T = T_1) = 0$ without violating any necessary conditions of optimality.

7. Dependence of Switching Time on Force Levels and Valuation of Y Survivors.

We consider the case when $\delta < 1$. Then we saw for the Isbell-Marlow problem (see Appendices A and F) that an extremal policy might be to shift the concentration of all Y-fire from X_1 to X_2 before the annihilation of X_1.

Now, there are two subcases for entry to S_4. We consider the extremals for which

$$\phi^*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T - \tau_1 \\ 0 & \text{for } T - \tau_1 < t \leq T. \end{cases} \tag{17}$$

Then, let us say that an extremal leads to S_4 when we use the extremal control (17) and reach the terminal state $x_1(T) > 0$, $x_2(T) > 0$, $y\ (T) = 0$, $T \leq T_1$. We denote the domain of controllability for S_4 when (17) is used by $D(S_4)$ and the corresponding payoff by P_4. It is convenient to introduce the "backwards time"

τ defined by $\tau = T-t$. Then according to previous arguments (see Appendix A where $\tau_1(S_4)$ is the same as $\tau_1(C_5)$), we have

$$\tau_1(S_4) = \frac{1}{\sqrt{a_2b_2}} \cosh^{-1} z, \tag{18}$$

where

$$z = \frac{R-\delta}{R-1} . \tag{19}$$

Now for S_1: $x_1(T) > 0$, $y(T) > 0$, $T=T_1$,

we have

$$p_3(t=T) = r \tag{20}$$

Also, recalling (11), we have

$$v(\tau=0) = a_1p-a_2q, \tag{21}$$

so that $v(\tau=0) < 0$ when $\delta = a_1p/(a_2q) < 1$. Recalling (13), which may be written as $\frac{dv}{d\tau} > 0$, we see that as τ increases $v(\tau)$ must change in sign from negative to positive. Let us denote this backwards time by τ_1. Then by (11), we see that $\phi^*(\tau) = 0$ for $0 \leq \tau \leq \tau_1$. We may use this extremal control to integrate the adjoint equations backwards in time. We may use the results to obtain

$$v(\tau) = (a_1b_1-a_2b_2)\left\{\frac{q}{b_2} \cosh \sqrt{a_2b_2}\ \tau + \frac{r}{\sqrt{a_2b_2}} \sinh \sqrt{a_2b_2}\ \tau\right\} + a_1p - \frac{a_1b_1q}{b_2} . \tag{22}$$

The "backwards time" of the first switch in tactics, denoted by τ_1, is determined by $v(\tau=\tau_1) = 0$, and this readily yields

$$\tau_1(S_1) = \frac{1}{\sqrt{a_2 b_2}} \ln \left(\frac{z + \sqrt{z^2 + \alpha^2 - 1}}{1 + \alpha} \right), \tag{23}$$

where

$$\alpha = \frac{r}{q}\sqrt{\frac{b_2}{a_2}} \quad . \tag{24}$$

We next prove a useful lemma.

LEMMA 1: Assume that $\delta < 1$.

Then

$$\frac{\partial \tau_1}{\partial \alpha} < 0 \ , \tag{25}$$

where $\tau_1 = \tau_1(S_1)$.

PROOF:

A. We readily compute from (23)

$$\frac{\partial \tau_1}{\partial \alpha} = \frac{\alpha + 1 - z^2 - z\sqrt{z^2+\alpha^2-1}}{\sqrt{a_2 b_2}\,(1+\alpha)\sqrt{z^2+\alpha^2-1}\,(z+\sqrt{z^2+\alpha^2-1}\,)} \tag{26}$$

B. Now $\delta > 1$ implies that $z > 1$, so that

$$\alpha < z\alpha < z\sqrt{z^2+\alpha^2-1} \ ,$$

whence $\quad \alpha + 1 - z^2 - z\sqrt{z^2+\alpha^2-1} < 0,$

which proves the lemma. Q.E.D.

Now by Lemma 1 we readily see that

$$\frac{\partial \tau_1}{\partial r} = \frac{\partial \tau_1}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial r} < 0 \ , \tag{27}$$

since $\frac{\partial \alpha}{\partial r} = \frac{1}{q}\sqrt{\frac{b_2}{a_2}} > 0$. Furthermore, it is easily shown that

$$\lim_{r \to +\infty} \tau_1(S_1) = 0. \tag{28}$$

Thus for S_1, if we value the survival of our own forces high enough, we never fire at X_2 when $x_1(t) > 0$ even though $a_1 p < a_2 q$. (To be precise, when $\delta < 1$, we can make the time

interval $[T - \tau_1, T]$ during which we concentrate all fire on X_2 arbitrarily small be taking r large enough.)

Let us now show that when the optimal policy is to change the allocation of fire (all concentrated on one target type) without a target-type force level being driven to zero, we fire at X_2 longer when $y(T) = 0$ than when $y(T) > 0$ (assuming that $R > 1$ and $\delta < 1$). This is the significance of the theorem that we now state and prove as Theorem 1.

THEOREM 1: Assume that $R > 1$ and $\delta < 1$.

Then,

$$\tau_1(S_1) < \tau_1(S_4).$$

PROOF:

A. Let us observe that both (18) and (23) may be considered to be special cases of the general equation

$$\tau_1 = \frac{1}{\sqrt{a_2 b_2}} \ \ell n \left(\frac{z + \sqrt{z^2 + a^2 - 1}}{1 + a} \right), \tag{29}$$

where

$$a = \frac{(r + \nu)}{q} \sqrt{\frac{b_2}{a_2}} \ . \tag{30}$$

We note that

(a) for $y(T) > 0$ (i.e. for S_1),

$\nu = 0,$

(b) for $y(T) = 0$ and $T < T_1$ (i.e. for S_4),

$\nu = -r$

B. Clearly, we have by (23) and (29)

$$\frac{\partial \tau_1}{\partial \nu} = \frac{\partial \tau_1}{\partial a}\frac{\partial a}{\partial \nu}. \tag{31}$$

C. Observing that $\frac{\partial \tau_1}{\partial \nu} = \frac{\partial \tau_1}{\partial a}\ \frac{\partial a}{\partial \nu}$ and $\frac{\partial a}{\partial \nu} = \frac{1}{q}\sqrt{\frac{b_2}{a_2}}$ > 0, the theorem readily follows by (31) and Lemma 1, since $- r = \nu(S_4) < \nu\ (S_1) = 0.$ Q.E.D.

Thus, we see that $y(T)$ affects the length of time that Y fires at X_2 when there is a change in the optimal tactic (concentration of all fire) before annihilation of X_1. Intuitively, we see that firing longer at X_1 prolongs the length of battle for those cases when $y(T) = 0$, since $a_1b_1 > a_2b_2$. We will prove this precisely in the next section and show that this is not an optimal tactic.

8. The Case When $T = T_1$ for S_4.

In the previous section we conjectured that when $R > 1$ and $\delta < 1$, we could prolong the length of battle leading to S_4 by firing longer at X_1. Let us consider the following. Assume that we have $P^o = (x_1^o, x_2^o, y_0)$ given and $\delta < 1$. Further assuming that P^o is such that an extremal leads to S_4 with $T(S_4) < T_1$, we recall that $p_3(t{=}T) = 0$. We will show that by adjusting the value

of $p_3(t=T) = \mu > 0$, we can steer the system to $y(T) = 0$ and $T = T_1$ (i.e. we can prolong the length of battle by the valuation placed on our forces). Finally, we will show that this can never be an optimal policy in order to drop this special case from further consideration (as we have done in Tables III).

We again consider the extremal control (17) which leads to S_4: $x_1(T) > 0, x_2(T) > 0, y(T) = 0, T \leq T_1$. In the special case when $T = T_1$, the transversality condition no longer holds so that $p_3(t=T) = \nu_3$ where ν_3 is unrestricted. Let us write $\nu_3 = r + \nu$ so that

$$\tau_1 = \tau_1(a) = \frac{1}{\sqrt{a_2 b_2}} \ln\left(\frac{z + \sqrt{z^2 + a^2 - 1}}{1 + a}\right), \tag{32}$$

$$a = \frac{(r+\nu)}{q}\sqrt{\frac{b_2}{a_2}} \, . \tag{33}$$

Using the extremal control (17), we readily obtain from the state equations (1)

$$y(T-\tau_1) = y_0 \cosh \sqrt{a_1 b_1}(T-\tau_1) - \frac{s}{\sqrt{a_1 b_1}} \sinh \sqrt{a_1 b_1}\,(T-\tau_1), \tag{34}$$

$$b_1 x_1(T-\tau_1) + b_2 x_2^o = s \cosh \sqrt{a_1 b_1}\,(T-\tau_1) - \sqrt{a_1 b_1}\, y_0 \sinh \sqrt{a_1 b_1}(T-\tau_1), \tag{35}$$

and

$$y(T) = y(T-\tau_1) \cosh \sqrt{a_2 b_2}\;\tau_1 - \frac{(b_1 x_1(T-\tau_1) + b_2 x_2^o)}{\sqrt{a_2 b_2}} \sinh \sqrt{a_2 b_2}\;\tau_1 . \tag{36}$$

(34), (35), and (36) are readily combined to yield

$$\frac{\partial y(T)}{\partial T} = -\left(b_1 x_1(T-\tau_1) + b_2 x_2^o\right)\cosh\sqrt{a_2 b_2}\,\tau_1 + R\sqrt{a_2 b_2}\,y(T-\tau_1)\sinh\sqrt{a_2 b_2}\,\tau_1. \tag{37}$$

Now T is defined by

$$y(T) = 0. \tag{38}$$

Combining (36) and (38) with (37), we obtain

$$\frac{\partial y(T)}{\partial T} = R\sqrt{a_2 b_2}\,y(T-\tau_1)\sinh\sqrt{a_2 b_2}\,\tau_1\left\{1 - \frac{\left(b_1 x_1(T-\tau_1) + b_2 x_2^o\right)^2}{a_1 b_1 y^2(T-\tau_1)}\right\}. \tag{39}$$

Let us recall the "generalized square law"

for $\phi(t) =$ <u>constant</u> for all $t \in [t_1, t_2]$

$$\zeta^2(t=t_1) - \zeta^2(t=t_2) = \left\{\phi a_1 b_1 + (1-\phi) a_2 b_2\right\}\left\{y^2(t=t_1) - y^2(t=t_2)\right\}, \tag{40}$$

where $\zeta(t) = b_1 x_1(t) + b_2 x_2(t)$. Now (17) and (40) yield

$$s^2 - a_1 b_1 y_0{}^2 = \left(b_1 x_1(T-\tau_1) + b_2 x_2^o\right)^2 - a_1 b_1 y^2(T-\tau_1). \tag{41}$$

Thus, (39) and (41) yield the following lemma

LEMMA 2: Consider $y(T)$ given by (36) and T defined by $y(T) = 0$. Then

$\frac{\partial y(T)}{\partial T} < 0$ if and only if

$a_1 b_1 y_0{}^2 < s^2$.

It is convenient to obtain the following from a backwards integration of the state equation of the state equations (1) using (17)

$$b_2x_2^o = b_2x_2(T)w + b_1x_1(T-\tau_1)(w-1). \tag{42}$$

We may then combine (36),(38), (17), and (40) to obtain

$$y^2(T-\tau_1) = \frac{(w^2-1)(s^2-a_1b_1y_0{}^2)}{a_1b_1 - w^2(a_1b_1-a_2b_2)} , \tag{43}$$

and further analysis yields

$$b_1x_1(T-\tau_1) = -b_2x_2^o + w\sqrt{\frac{s^2 - a_1b_1y_0{}^2}{R - w^2(R-1)}} , \tag{44}$$

and

$$b_2x_2(T) = b_2x_2^o + (1-w)\sqrt{\frac{s^2 - a_1b_1y_0{}^2}{R - w^2(R-1)}} , \tag{45}$$

where

$$w = w(a) = \cosh\sqrt{a_2b_2}\ \tau_1(a) . \tag{46}$$

Now, we may use (43) to solve (34) for $T - \tau_1$ to obtain

$$T - \tau_1 = \frac{1}{2\sqrt{a_1b_1}}\ \ell n\left\{\frac{(w - \sqrt{R(w^2-1)})(s + \sqrt{a_1b_1}\ y_0)}{(s - \sqrt{a_1b_1}\ y_0)(w+ \sqrt{R(w^2-1)})}\right\} \tag{47}$$

Combining (32) and (46), we obtain

$$w(a) = \begin{cases} \dfrac{z - a\sqrt{z^2 + a^2 - 1}}{1 - a^2} & \text{for } 0 \le a < 1, \\[2ex] \dfrac{z^2 + 1}{2z} & \text{for } a = 1, \\[2ex] \dfrac{a\sqrt{z^2 + a^2 - 1} - z}{a^2 - 1} & \text{for } a > 1. \end{cases} \qquad (48)$$

Differentiation of (48) yields

$$\frac{\partial w}{\partial a} = \frac{-(z\alpha - \sqrt{z^2 + a^2 - 1})^2}{\sqrt{z^2 + a^2 - 1}\ (1-a^2)^2} \qquad \begin{array}{l} \text{for } 0 \le a < 1 \\ \text{and } a > 1, \end{array} \qquad (49)$$

and hence

$$\frac{\partial w}{\partial a} < 0 \quad \text{for} \quad 0 \le a < 1 \quad \text{and} \quad a > 1. \qquad (50)$$

Considering (47), it is easily seen that for $P^o \in D(S_4)$ with $T = T_1$ and $a_1 b_1 y_0{}^2 < s^2$, we must have

$$1 \le w < \sqrt{\frac{R}{R - 1}}\ . \qquad (51)$$

Since T is implicitly determined by (38), which we re-write as

$$y(\tau_1,\ T) = 0, \qquad (52)$$

partial differentiation readily leads to

$$\frac{\partial T}{\partial \nu} = \frac{-\dfrac{\partial y}{\partial \tau_1} \cdot \dfrac{\partial \tau_1}{\partial \nu}}{\dfrac{\partial y}{\partial T}}\ . \qquad (53)$$

Now, $\dfrac{\partial\tau_1}{\partial\nu} = \dfrac{\partial\tau_1}{\partial a}\ \dfrac{\partial a}{\partial\nu} = \dfrac{\partial\tau_1}{\partial\alpha}\ \dfrac{\partial a}{\partial\nu}$, or using (33)

$$\frac{\partial\tau_1}{\partial\nu} = \frac{1}{q}\sqrt{\frac{b_2}{a_2}}\ \frac{\partial\tau_1}{\partial\alpha} < 0, \tag{54}$$

where we recall Lemma 1 (see (25)).

Now, partial differentiation of (36) and use of (34) and (35)(including partial differentiation of (34) and (35) with respect to τ_1) yields

$$\frac{\partial y(T)}{\partial\tau_1} = -\ (R-1)\sqrt{a_2b_2}\ y\ (T-\tau_1)\ \sinh\sqrt{a_2b_2}\ \tau_1, \tag{55}$$

so that

$$\frac{\partial y(T)}{\partial\tau_1} < 0 \quad \text{for} \quad R > 1. \tag{56}$$

We now state

THEOREM 2: Assume that $R > 1$, $\delta < 1$, (56)

$a_1b_1y_0{}^2 < s^2$, and $1 \le w < \sqrt{\dfrac{R}{R-1}}$.

Then

$$\frac{\partial T}{\partial\nu} > 0.$$

PROOF: Immediate by (51), (53), (54), (56), and Lemma 2. Q.E.D.

Thus, we see that by increasing the implicit valuation of Y-forces (i.e. ν) the length of battle can be extended until $T = T_1$. To sum up for S_4: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$, $T \le T_1$, in the special case when $T = T_1$ we have so far found no violation of any necessary condition of optimality in having $-r \le \nu$. Then there is a region in the initial state (force level) space from

which extremals lead to this special end condition, and in this region extremals also lead to S_4 with $T < T_1$ (i.e. recall that for such P^o $T < T_1$ when we take $p_3(t-T)=0$).

We now show, however, that although the length of battle can be extended so that $y(T=T_1)= 0$, this results in a reduced payoff. Recalling that we denote the payoff associated with the extremal control (17) by P_4, we can use (44) and (45) to obtain

$$P_4 = \frac{q}{b_2}\left\{-z\left(\frac{R-1}{R}\right) b_2 x_2^o - \left(\frac{R - wz(R-1)}{R}\right)\sqrt{\frac{s^2 - a_1 b_1 y_0{}^2}{R - w^2(R-1)}}\right\} . \tag{57}$$

Differentiation of (57) and use of (33) readily yields

$$\frac{\partial P_4}{\partial \nu} = \frac{(R-1)}{\sqrt{a_2 b_2}} \frac{\partial w}{\partial a} \frac{(s^2-a_1 b_1 y_0{}^2)^{1/2}}{(R-w^2(R-1))^{3/2}} (z-w). \tag{58}$$

Considering (48) and (50), we have

$$w(a=0) = z \quad \text{and} \quad \frac{\partial w}{\partial a} < 0 \quad \text{for} \quad 0 \le a < 1 \quad \text{and} \quad a > 1. \tag{59}$$

Hence,

$$w(a) < z \quad \text{for} \quad 0 < a . \tag{60}$$

Thus, (50), (58), and (60) yield

THEOREM 3: Assume that $R > 1$, $\delta < 1$,

$a_1 b_1 y_0{}^2 < s^2$, and $1 \le w < \sqrt{\frac{R}{R-1}}$.

Then

$$\frac{\partial P_4}{\partial \nu} < 0.$$

9. Discussion.

There are two topics that especially merit discussion: (1) the structure of the optimal target selection policy and (2) the development of solutions to such problems. We also suggest a future research task.

From our work reported here in Appendix G on the prescribed duration battle, we conclude that the structure of the optimal target selection policy may depend (indirectly) on (initial) force levels. This revises a tentative conclusion stated in our earlier work [3] (see also Appendix B). To elaborate further, if no target type is annihilated, then there is no explicit dependence of the optimal target selection policy on force levels. However, the non-annihilation of all target types itself depends upon the initial force levels. Furthermore, we saw that for $\delta = a_1 p/(a_2 q) < 1$ the optimal length of time that Y fires at X_2 depends on whether or not $y(T) = 0$. Moreover, for $\delta \geq 1$ not only does the optimal target selection policy take an extremely simple form (concentrate all fire on X_1 while $x_1 > 0$) but it also is independent of the force levels. When $\delta < 1$, the solution was extremely complex except for the case when Y annihilates both X_1 and X_2 in which case the same simple optimal policy as when $\delta \geq 1$ applies.

It seems appropriate to compare the structure of the optimal target selection policy for the prescribed duration battle with that for the terminal control (fight-to-the-finish) battle. We

have seen that we may consider the fight to the finish to be imbedded in the prescribed duration battle. For such cases when $T < T_1$, the solutions of the two problems are identical. Furthermore, when $\delta \geq 1$ the structures of the optimal policies were identical. We conjecture that this will always be true for n-versus-one combat when surviving target types are valued in direct proportion to their kill capabilities. However, if the initial force levels are such that the specified maximum duration for the battle (denoted by T_1) terminates the battle, then the structures were different when $\delta < 1$. Thus, we see that even the nature of the scenario for a target selection problem may have an effect upon the optimal policy.

In developing a (hopefully) complete solution for the prescribed duration battle, we have encountered some novel and interesting features that require further discussion. First of all, it does not seem possible to develop an entirely analytic solution to such problems (see Tables III). Computational methods must be combined with analytic results arising from theoretical optimality conditions in order to determine the optimal policy. It should be noted that difficulties (multiple extremals from an initial point P^o) arise in the n-versus-one battle when target types are not valued in direct proportion to their kill rates. In the second place, the theory of state variable inequality constraints (SVIC) must be used to solve such problems. We have

already pointed this out at several places elsewhere in this report (see Appendixes E and F). Thirdly, in all the problems that we have studied in the same depth as for the problem at hand, in developing a complete solution we have invariably come across some special cases that require an inordinate amount of analysis. For this prescribed duration battle, one such case was for S_4 when $T = T_1$. However, such special cases require treatment in order that a complete solution be obtained.

Let us point out that for the simplest (deterministic) optimal control problems of target selection that we have now studied a tremendous amount of analysis has been required to develop a complete solution. We do not believe that any other type of model or formulation (for example, discrete-time formulations solved by nonlinear programming or the formalism of dynamic programming) would reduce any of the key difficulties which have required the majority of the analysis. We have, for example, considered a discrete-time version of the Isbell-Marlow problem (see Appendices A and F) and used nonlinear programming (Kuhn-Tucker conditions) to develop the basic optimality conditions. Again, for the finite-dimensional problem, there is no way to circumvent the non-uniqueness of extremal paths leading to different terminal states. Similar remarks apply when one uses the formalism of dynamic programming.

One task remains to be done, and we propose this to ONR as a future research task: examination of the sensitivity of the payoff to the target selection policy. With our solutions to such problems (i.e. both terminal control and prescribed duration) available, it would seem desirable to examine whether or not a simple (probably non-optimal) target selection policy might yield nearly the same return as the true optimal policy. For example, for the prescribed duration battle we have seen that the solution is amazingly complex when $\delta < 1$. In fact, we can't explicitly state the optimal policy in general but can only numerically compute the optimal policy for a given set of input values. A simple target selection policy which would yield close to the optimal return would clearly be of great value in this case.

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Appendix H. On the Solution to Lanchester-Type Equations of Modern Warfare with Variable Coefficients.

1. Introduction.

This appendix develops solutions to extensions of F. W. Lanchester's classical equations of modern warfare (frequently referred to as aimed-fire equations) for combat between two homogeneous forces. In these extensions the lethality of the fire (as expressed by the Lanchester attrition-rate coefficient) depends upon time. When the dependence is arbitrary, the solution is an infinite series of recursively related integrals; in special cases, more convenient representations (including representation in terms of tabulated functions) are available. Solutions are obtained in the following cases: (1) lethality of each side's fire proportional to a power of time and both lethalities initially zero, and (2) lethality of each side's fire linear with time but only one side's lethality initially zero. The latter case models the constant speed approach between forces whose weapons have different maximum effective ranges.

In a previous note published in the open literature [33] we developed a solution to variable-coefficient Lanchester-type

equations of modern warfare (various Lanchester-type equations of warfare are reviewed and our terminology explained below) for combat between two homogeneous forces in the special case of a constant ratio of Lanchester attrition-rate coefficients. A sketch of some previous work on variable-coefficient Lanchester-type equations is to be found there. In the present paper we develop solutions to extensions of F. W. Lanchester's classical equations of modern warfare in which the lethality (or effectiveness) of fire (as expressed by the Lanchester attrition-rate coefficient) may change over time. Our results point out the complexities of analytically treating such extensions of Lanchester's classical work (see [24]).

Our present work considers only the purely formal, mathematical aspects of the Lanchester theory of combat. We hope that others will continue to address the equally important scientific questions as to whether such hypotheses as considered here are tenable in the light of empirical evidence (see, for example, [17], [18], [34], [35], [37]). We must also note that S. Bonder's excellent work [4], [6] on the Lanchester attrition-rate coefficient (see also [2], [22]) very naturally leads to the requirement of having solutions to variable-coefficient Lanchester-type equations available to analysts in force-structuring and weapons system design studies. The results of this

appendix are hopefully a step in that direction. Other recent papers have discussed the estimation of parameters for the Lanchester attrition-rate coefficient [30], [31].

Similar approaches towards variable-coefficient Lanchester-type equations have been previously used by T. Oberbeck [28] and F. Dashiell and W. Fain [9]. A summary of Oberbeck's work and other early work on Lanchester-type equations is to be found in the summary report by R. Snow [32]. Dashiell and Fain [9] base their work on a theoretical paper by Whyburn [36], but do not cite Oberbeck's pioneering work or any of the numerous excellent applied mathematical treatments of systems of first order ordinary differential equations (see, for example, pp. 53-55 of [15] or pp. 73-77 of [23]). Although Whyburn's method of successive approximations is applicable to systems of variable coefficient differential equations, Dashiell and Fain consider only the constant coefficient case. This work is the mathematical basis of later tactical simulation studies using matrix Lanchester-type equations [7], [11].

In this appendix we first review various Lanchester-type equations of warfare. Then we show a simpler derivation of our previous result for a constant ratio of attrition-rate coefficients. This development is later used for insight into cases when an infinite series may be represented by tabulated functions. Next, we develop a general solution to variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces in the form of an

infinite series of recursively related terms by considering an equivalent integral equation. We then consider some special types of attrition-rate coefficients for which this series reduces to tabulated functions and an important case for which it does not. Several numerical examples are given. Finally, we discuss the significance, applications, and extensions of our results.

2. Some Lanchester-Type Equations of Warfare.

It seems appropriate to briefly review some Lanchester-type equations that have appeared in the literature. In our discussion here, we will consider equations with attrition-rate coefficients that depend upon time, although in the original sources cited these were assumed to be constant.

In 1914 in an article published in the British journal Engineering (see [24]), F. W. Lanchester hypothesized that combat between two opposing forces "under modern conditions" could be modelled by (see [34] for a discussion of the assumptions inherent in (1) with constant lethality of fire)

$$\frac{dx}{dt} = -\, a(t)y,$$

$$\frac{dy}{dt} = -\, b(t)x, \tag{1}$$

where $x(t)$ and $y(t)$ refer to the numbers of X- and Y-forces, respectively. The functions $a(t)$ and $b(t)$ are usually termed [2], [6], [22] the Lanchester attrition-rate coefficients and represent

the lethality of each side's fire. We shall refer to equations (1) as Lanchester's equations of modern warfare. These equations are appropriate when both sides use aimed fire and target acquisition times are negligible [8], [34].

This paper concerns the solution of equations (1). However, other differential equation combat models may be referred to as Lanchester-type equations. Although not explicitly given by Lanchester [24], the following equations were hypothesized to describe hand-to-hand combat (one against one) in "ancient warfare"

$$\frac{dx}{dt} = - a(t),$$

$$\frac{dy}{dt} = - b(t). \tag{2}$$

Lanchester did give [24] the following equations for area fire (see [34] for discussion of assumptions)

$$\frac{dx}{dt} = - a(t)xy,$$

$$\frac{dy}{dt} = - b(t)xy. \tag{3}$$

Equations (3) are also appropriate for aimed fire when target acquisition time is much larger than the time to destroy an acquired target and target acquisition time is inversely proportional to target density [8]. More recently, S. Deitchman [10] used the following equations to model ambush situations (in guerrilla warfare) in which the X-forces (ambushers) use aimed fire against ambushees who return area fire

$$\frac{dx}{dt} = - a(t)xy,$$

$$\frac{dy}{dt} = - b(t)x. \tag{4}$$

The a's and b's (Lanchester attrition-rate coefficients) may be, of course, different in equations (1) through (4) and related to different physical quantities. Other factors such as replacements, operational losses, effects of supporting fires, etc., may be accounted for by appropriate modifications of the above equations, but we will not consider these.

The above differential equation models of combat are deterministic, always yielding the same result for given initial conditions. However, such attrition equations are commonly assumed to represent the mean course of battle (implying an underlying probability distribution [32]). Other than remarking that this is why we will refer to $x(t)$ as the average X-force level (this is well-known to be only an approximation [32]), we will not discuss stochastic aspects further.

3. Constant Ratio of Attrition-Rate Coefficients: Alternate Development 1.

We derive a previously obtained result [33] in such a manner as to shed light on the circumstances under which our general result of the next section reduces to tabulated functions. Let us consider equations (1). We have previously shown that it is inessential whether we consider time or force separation (range) as the independent variable

(see [33] for a transformation to change a solution with time as the independent variable to one with range as the independent variable).

We may combine the two equations of (1) into the following single equation for $x(t)$

$$\frac{d}{dt}\left\{\frac{1}{a(t)}\frac{dx}{dt}\right\} - b(t)x = 0, \tag{5}$$

with initial conditions

$$x(t=0) = x_0 \quad \text{and} \quad \left[\frac{1}{a(t)}\frac{dx}{dt}\right]_{t=0} = -y_0 .$$

Introducing a new independent variable $u = \int_0^t a(s)ds$, we may transform equation (5) into

$$\frac{d^2x}{du^2} - \frac{b(t)}{a(t)}x = 0 . \tag{6}$$

We note that we require the integral $\int_0^t a(s)ds$ to exist in order for this transformation to be applicable. At this point, the form of equation (6) makes our previous result obvious. It also is helpful in obtaining solutions and approximations in other cases. Thus when $b(t)/a(t) = \text{constant}$ or $a(t) = k_a h(t)$ and $b(t) = k_b h(t)$, we have

$$\frac{d^2x}{du^2} - \frac{k_b}{k_a}x = 0,$$

with

$$x(u=0) = x_0,$$

$$\frac{dx}{du}(u=0) = -y_0,$$

so that

$$x(u) = x_0 \cosh\sqrt{k_b/k_a}\,u - y_0\sqrt{k_a/k_b}\sinh\sqrt{k_b/k_a}\,u.$$

Hence

$$x(t) = x_0 \cosh \theta(t) - y_0\sqrt{k_a/k_b} \sinh \theta(t), \tag{7}$$

where $\theta(t) = \sqrt{k_a k_b} \int_0^t h(s)ds$ and a similar expression may be obtained for $y(t)$.

There is, however, a much greater significance of equation (6). What we really have is

$$\frac{d^2x}{du^2} + I(u)x = 0, \tag{8}$$

where $I(u) = -b(t)/a(t)$ and $u = \int_0^t a(s)ds$.
Equation (8) is the "normal form" of equation (5) (after Felix Klein, see p. 158 of [13]). There is an infinite series solution in terms of repeated integrations (first developed by Lord Kelvin, see p. 120 of [14]) available for equation (8) (also see p. 351 of [29]). We shall not pursue this approach further here, however.

The term $I(u)$ is called the "invariant" of the normal form. Richards (pp. 347-350 of [29]) has developed a table of various functions, $I(u)$, for second-order linear differential equations written in the normal form. He claims that over 180 of the first 220 second-order linear equations listed by Kamke [21] can immediately be solved by use of this table. Kamke's [21] classic work contains among other things a tabulation of solutions to second-order linear differential equations, similar to a table of integrals. This means that if $I(u)$ does not take a form contained in Richards' tables, then it is extremely unlikely that equation (8) has a solution expressible by any of the standard functions of analysis and that some type of infinite series is the best that we can hope for.

4. Arbitrary Attrition-Rate Coefficients: Integral Equation Approach.

In this section we derive a general solution to equations (1) by means of an infinite series of terms with repeated integrations. Our approach is, of course, formally equivalent to approaches mentioned earlier (see pp. 53-56 of [15] or pp. 73-77 of [23]).

By repeated integrations, we may obtain the following Volterra integral equation from equations (1)

$$x(t) = x_0 - y_0 \int_0^t a(s_1)ds_1 + \int_0^t a(s_1)ds_1 \int_0^{s_1} b(s_2)x(s_2)ds_2. \qquad (9)$$

Equation (9) is readily solved by Picard's method of successive approximations (see p. 109 of [12] or pp. 61-74 of [16]) to yield

$$x(t) = x_0 f(t) - y_0 \int_0^t a(s)g(s)ds, \qquad (10)$$

where

$$f(t) = \sum_{n=0}^{\infty} H_n(t), \quad g(t) = \sum_{n=0}^{\infty} K_n(t), \qquad (11)$$

and

$$H_0(t) = 1, \quad K_0(t) = 1,$$

for $n > 0$

$$H_n(t) = \int_0^t a(s_1)ds_1 \int_0^{s_1} b(s_2)H_{n-1}(s_2)ds_2,$$

$$K_n(t) = \int_0^t b(s_1)ds_1 \int_0^{s_1} a(s_2)K_{n-1}(s_2)ds_2. \qquad (12)$$

The formal steps above are justified, for example, when $a(t)$ and $b(t)$ are bounded for $t \in [0,T]$, since $f(t)$ is then majorized by $\cosh\sqrt{AB}\, t$ where $A = \max_{t \in [0,T]} a(t)$, etc. Under these circumstances it is easy to show the convergence to this method of successive approximations (see pp. 113-114 of [12] or pp. 61-64 of [16]), and that $f(t)$ is an absolutely and uniformly convergent series for $t \in [0,T]$.

Equations (10), (11) and (12) are a general solution to the Lanchester-type equations (1). Being general, this solution is complex but still is in a form amenable to computer-aided computation. An infinite series solution, however, provides little insight into the effect on battle outcome of attrition rate parameters, short of our generating parametric curves by grinding out solutions for specific parameter values. A computer may not always be available to aid in the summing of the infinite series. Hence, it is desirable to know when the series solution (10) may be expressed in terms of tabulated functions. In the next two sections we examine two cases of when this is possible. We omit verification that (10) reduces to the well-known constant coefficient solution when $a(t)$ and $b(t)$ are constants, since this has already been done by Dashiell and Fain (see pp. 94-95 of [9]), whose approach may be shown to be equivalent to that given here.

5. Constant Ratio of Attrition-Rate Coefficients: Alternate Development 2.

We show that (10) reduces to our previous result (7) when $a(t) = k_a h(t)$ and $b(t) = k_b h(t)$. This present derivation plus our first alternate are useful in providing insight into the circumstances under which (1) possesses exponential solutions.

We develop the first term, $f(t) = \sum_{n=0}^{\infty} H_n(t)$, in equation (10). From our assumed conditions, we have that

$$H_1(t) = k_a k_b \int_0^t h(s_1)ds_1 \int_0^{s_1} h(s_2)ds_2,$$

which by a differentiation and subsequent integration may be shown to be equal to

$$H_1(t) = \frac{k_a k_b}{2} \left[\int_0^t h(s)ds\right]^2 .$$

Using the fact that

$$h(s_1)\left[\int_0^{s_1} h(s_2)ds_2\right]^n = \frac{1}{(n+1)} \frac{d}{ds_1}\left\{\left[\int_0^{s_1} h(s_2)ds_2\right]^{(n+1)}\right\},$$

we can easily establish inductively that

$$H_n(t) = \frac{(k_a k_b)^n}{(2n)!} \left[\int_0^t h(s)ds\right]^{2n}, \tag{13}$$

so that $f(t) = \cosh\{\sqrt{k_a k_b} \int_0^t h(s)ds\}$, where we have made use of the power series expansion for $\cosh x$. A similar expression is readily obtained for $g(t)$, and thus (7) follows from (10), (11) and (12) in this special case. However, we now see that the simple "exponential" solution (7) depends on $H_n(t)$ being given by (13). This apparently will occur only when $a(t)/b(t) = \text{constant}$. Thus, this is the only instance that we have encountered so far in which variable-coefficient Lanchester-type equations of modern warfare have such a simple solution.

6. Another Example: Power Attrition-Rate Coefficients.

A case of probably greater practical interest is when the attrition-rate coefficients in (1) take the following form

$$a(t) = k_a t^m,$$

$$b(t) = k_b t^n, \tag{14}$$

where we will refer to m and n as the exponents of the power attrition-rate coefficients. We shall sketch the development of the solution via the Volterra integral equation as given by (10), (11) and (12).

We focus on the development of $f(t)$ in (10). Combining (12) and (14), we find that

$$H_1(t) = k_a k_b \int_0^t s_1^m ds_1 \int_0^{s_1} s_2^n ds_2 = \frac{k_a k_b t^{m+n+2}}{(n+1)(m+n+2)},$$

where we must place the following restrictions on m and n: $m > -1$ and $n > -1$. It is readily established by mathematical induction that

$$H_k(t) = \frac{(k_a k_b)^k \, t^{k(m+n+2)}}{k!(m+n+2)^k \prod_{i=0}^{k-1} \{n+1+i(m+n+2)\}} . \tag{15}$$

Equation (15) may be rearranged to yield

$$H_k(t) = \Gamma(1-p) \left(\frac{\sqrt{k_a k_b}}{2s}\right)^{2k} \frac{t^{2ks}}{k!\Gamma(k+1-p)} , \tag{16}$$

and hence,

$$f(t) = \sum_{k=0}^{\infty} H_k(t) = \Gamma(1-p) \left(\frac{\sqrt{k_a k_b}}{2s}\right)^{p} t^{ps} I_{-p}\left(\frac{\sqrt{k_a k_b}}{s} t^s\right) , \tag{17}$$

where

$$s = (m+n+2)/2, \qquad p = (m+1)/(m+n+2),$$

and $I_\nu(x)$ is the modified Bessel function of the first kind and order ν. It has the following power series expansion

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\nu+2n}}{n!\Gamma(n+1+\nu)} .$$

A similar result is to be obtained for $g(t)$. Thus, we obtain the following expression for the average strength of the X-forces

$$x(t) = t^{\frac{m+1}{2}} \left\{ x_0 \Gamma(1-p) \left(\frac{\sqrt{k_a k_b}}{2s}\right)^{p} I_{-p}\left(\frac{\sqrt{k_a k_b}}{s} t^s\right) - \frac{k_a y_0 \Gamma(1+p)}{(m+1)} \left(\frac{\sqrt{k_a k_b}}{2s}\right)^{-p} I_{p}\left(\frac{\sqrt{k_a k_b}}{s} t^s\right) \right\} . \tag{18}$$

Since few Bessel functions of fractional order are tabulated, we also observe that this solution (18) is also equivalent to the following power series

$$x(t) = x_0\Gamma(1-p)\left\{\sum_{k=0}^{\infty}\left[\frac{\sqrt{k_a k_b}}{2s}\right]^{2k}\frac{t^{2ks}}{k!\,\Gamma(k+1-p)}\right\}$$

$$-\frac{k_a y_0\Gamma(1+p)}{(m+1)}\left\{\sum_{k=0}^{\infty}\left[\frac{\sqrt{k_a k_b}}{2s}\right]^{2k}\frac{t^{2(ks+ps)}}{k!\,\Gamma(k+1+p)}\right\}. \qquad (19)$$

Similar expressions are obtainable for $y(t)$. The same results can be obtained by considering (8) and using Richards' table (see p. 349 of [29]). In this particular instance, probably the easiest way to obtain (18) and (19) is to observe the ordinary differential equations satisfied by Bessel functions (see pp. 166-167 in [19]).

When $m = n$, then $s = m + 1$ and $p = 1/2$ in (18). Observing that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sinh x, \quad \text{etc.} \quad \text{(see p. 165 of [19])}$$

and that

$$\Gamma(1/2) = \sqrt{\pi}\ ,$$

we can easily show that (18) reduces to (7) in this special case.

It is of interest to the operations research worker who desires numerical solutions that Lebedev (see p. 112 of [25]) notes that J. Liouville has shown that the case of half-integral order, i.e.,

$\nu = n + 1/2$ where n is an integer, is the only case where Bessel functions reduce to "elementary functions." Hence, except for the case $m = n$ it is not possible to express $x(t)$ as given by equation (18) in terms of elementary functions.

7. Weapon Systems with Different Effective Ranges: Linear Attrition-Rate Coefficients.

Consideration of the attrition-rate coefficients given by equations (14) leads us to realize that they apply only to combat between forces using weapons with the same effective range, i.e., attrition commencing for both sides at the same time. Consider the example of a constant speed attack of a mobile force against a static defense. Assuming that firing by either side commences when its enemy is within the effective range of the weapon system, we see that (14) has implicit within it the assumption that both forces achieve nonzero attrition rates at the same range. Of interest is the case where opposing weapon systems have different effective ranges. An example of time dependent attrition-rate coefficients which correspond to the lethality of each side's fire depending linearly upon range in such a scenario is given by

$$a(t) = k_a t,$$

$$b(t) = k_b (t+A). \qquad (20)$$

The range dependence of these Lanchester attrition-rate coefficients is depicted in Figure 1.

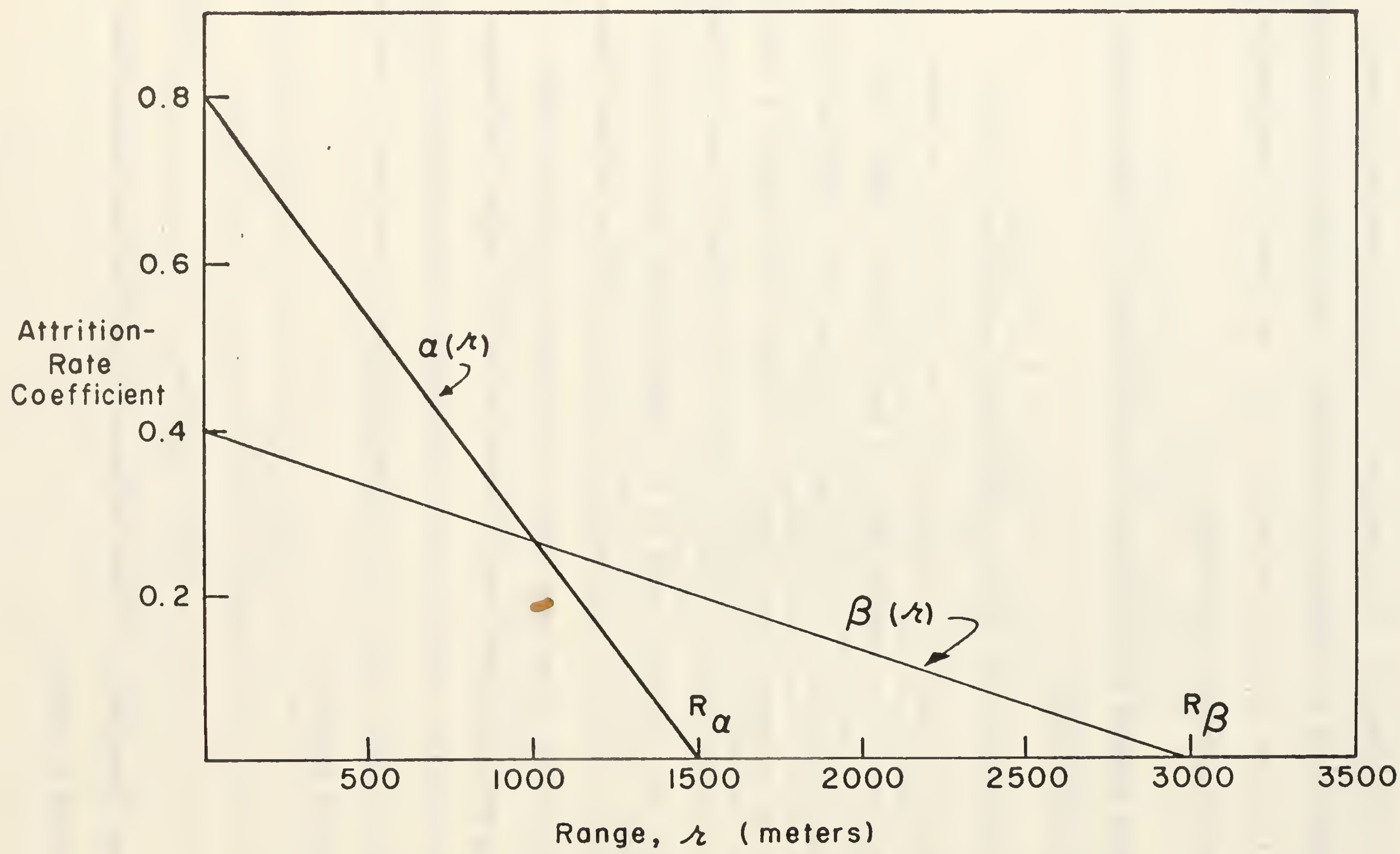


Figure 1. Linear attrition-rate coefficients for weapon systems of different effective ranges.

We become aware of the great generality of the solution as given by (10), (11) and (12) when the attrition-rate coefficients in equations (1) are given by (20). In this case, we have not been able to develop a solution in terms of tabulated functions via Richards' tables [29] after (1) has been converted to the normal form (8). However, the following solution is readily established inductively

$$x(t) = x_0 \left\{ \sum_{n=0}^{\infty} F_n(t) \right\} - y_0 \sqrt{\frac{k_a}{k_b}} \left\{ \sum_{n=0}^{\infty} G_n(t) \right\} \qquad (21)$$

where

$$F_n(t) = \left(\frac{\sqrt{k_a k_b}}{2} \right)^{2n} \frac{t^{3n}}{(2n)!} \sum_{k=0}^{n} B_n^k A^k t^{n-k},$$

$$B_0^0 = 1,$$

and for $n > 0$, we have

$$B_n^k = \begin{cases} 1 & \text{for } k = 0, \\ \left[\dfrac{4n(4n-2)}{(4n-k)(4n-k-2)} \right] \left(B_{n-1}^k + B_{n-1}^{k-1} \right) & \text{for } 1 \le k \le n-1, \\ \dfrac{4}{3} \dfrac{(4n-2)}{(3n-2)} B_{n-1}^{n-1} & \text{for } k = n, \end{cases}$$

and

$$G_n(t) = \left(\frac{\sqrt{k_a k_b}}{2} \right)^{2n+1} \frac{t^{3n+2}}{(2n+1)!} \sum_{k=0}^{n} C_n^k A^k t^{n-k},$$

$$C_0^0 = 1,$$

and for $n > 0$, we have

$$C_n^k = \begin{cases} 1 & \text{for } k = 0, \\ \left[\dfrac{4n(4n+2)}{(4n-k)(4n-k+2)} \right] \left(C_{n-1}^k + C_{n-1}^{k-1} \right) & \text{for } 1 \le k \le n-1, \\ \dfrac{4}{3} \dfrac{(4n+2)}{(3n+2)} C_{n-1}^{n-1} & \text{for } k = n. \end{cases}$$

A similar result is readily obtained for $y(t)$. It is easily seen that (21) reduces to our previous results when $A = 0$.

The form in which we have written this solution (21) is of particular value for numerical computations, since it is of the form (for $t > 0$)

$$x(t) = x_0 f(\lambda_1, \lambda_2) - y_0 \sqrt{\frac{k_a}{k_b}}\, g(\lambda_1, \lambda_2) ,$$

where λ_1 and λ_2 are parameters defined by

$$\lambda_1 = \sqrt{k_a k_b}\, \frac{t^2}{2} ,$$

$$\lambda_2 = \frac{A}{t} .$$

This allows us to develop tabulations of f and g, depending on the two parameters λ_1 and λ_2, which may be used to generate numerical solutions to all such problems as we have considered in this section. These functions are similar to $\cosh \lambda_1$ and $\sinh \lambda_1$ (depending on a single parameter), to which they reduce when $A = 0$, i.e., $\lambda_2 = 0$.

It is of interest to note that if the infinite series method of Frobenius (see pp. 78-97 of [13] or pp. 132-139 of [19]) is used to solve the above problem, we have not been able to obtain the solution in the form of (21), although the series so obtained is readily verified to be a rearrangement of (21). The absolute convergence of the series in question justifies this rearrangement. This shows that the solution

representation given by (10), (11) and (12) is a particularly important one, since it leads to results of practical value that we have not been able to obtain by any other means.

8. Some Numerical Examples.

In this section we examine two numerical examples. S. Bonder [3], [5] has pioneered in the study of range capabilities and mobility considerations for weapon systems in combat described by Lanchester-type equations of modern warfare. Our numerical results illustrate how his type of analysis may be extended to weapon systems with (1) different range dependencies of lethality of each side's fire (but the same effective range) and (2) linear attrition-rate coefficients but different effective ranges.

We consider a constant speed attack of a defended position with the combat described by

$$\frac{dx}{dt} = -\alpha(r)y = -\alpha_0\left(1 - \frac{r}{R_\alpha}\right)^m y,$$

$$\frac{dy}{dt} = -\beta(r)x = -\beta_0\left(1 - \frac{r}{R_\beta}\right)^n x, \tag{22}$$

where R_α and R_β are effective ranges of the Y- and X-force weapon systems, i.e. $\alpha(r) = 0$ for $r > R_\alpha$. Range is related to time by the relationship

$$r(t) = R_0 - vt \ ,$$

where R_0 is the opening range of battle. Several range dependencies for attrition-rate coefficients are illustrated in Figure 2.

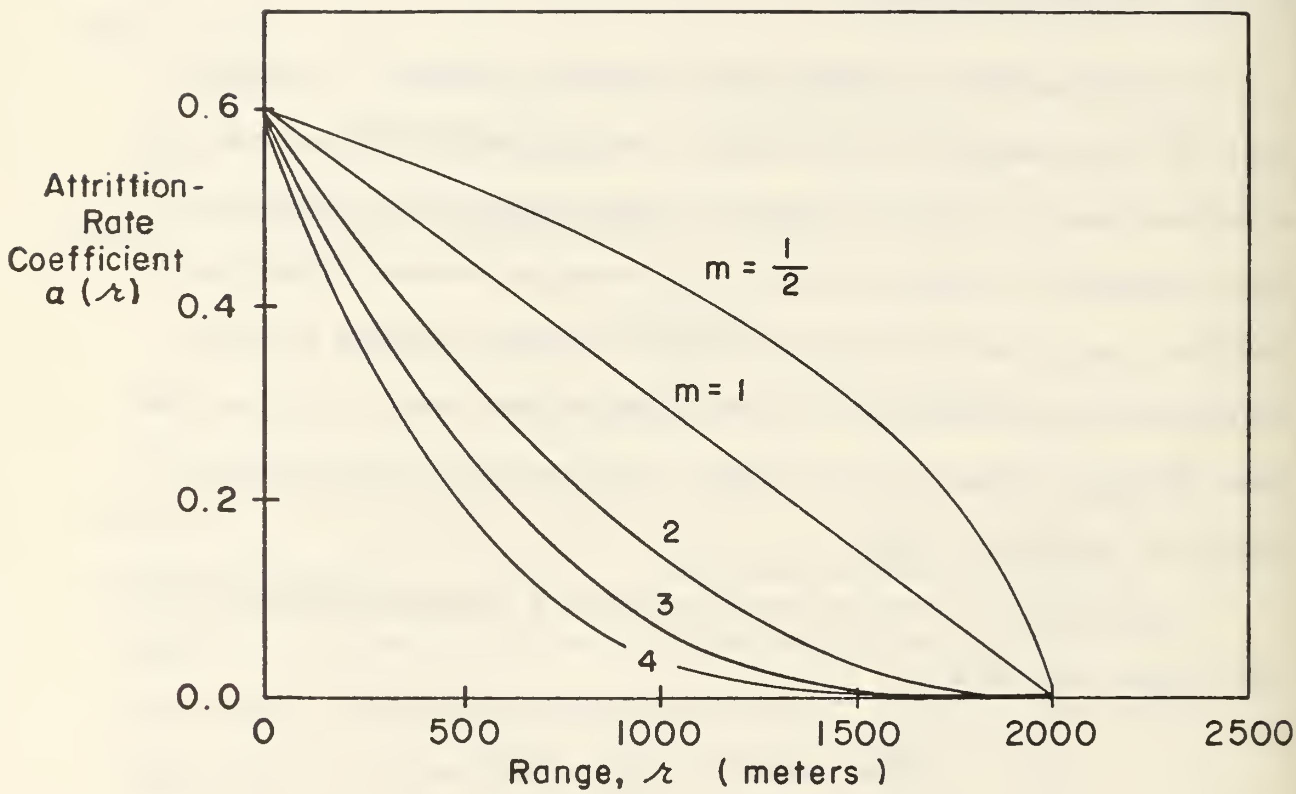


Figure 2. Dependence of attrition-rate coefficient, $a(r)$, upon exponent m.

Numerical results (after an example of S. Bonder, see Tab H, p. IV-34 of [5]) are shown in Figures 3, 4, 5 and 6. These curves have been generated by a computer program using the series solutions (19) and (21), as appropriate. To check the computer algorithm, the series solution (19) was compared with the solution given by (18) in terms of modified Bessel functions of the first kind and fractional order (it is easily seen that $0 < p < 1$). Available tables of modified Bessel functions are discussed in [1] (see pp. 455-456) and [26] (see p. 41). We made use of the National Bureau of Standards' tables [27] in our work. Although tables for I_ν only exist for $\nu = \pm 1/4$, $\pm 1/3$, $\pm 2/3$ and $\pm 3/4$ (see also p. 235 of [20]), interpolation methods may be used for other fractional values. In [27] Lagrange interpolation polynomial (8th degree) coefficients are given which yield four decimal place accuracy for $I_\nu(x)$ for $0 < \nu < 1$. The above considerations are useful when one does not have a computer readily available for computation.

In Figures 3 and 4 both systems have the same effective range, i.e., $R_\alpha = R_\beta$, and the battle commences at this range, i.e., $R_0 = R_\alpha$. In this plot we have held $\alpha_0 = \alpha(r=0)$ and β_0 constant and have varied the exponents, m and n, of the range dependencies of $\alpha(r)$ and $\beta(r)$, respectively. S. Bonder's pioneering work [3], [5] has extensively studied the cases of $m = n$. We see that the nature of the combat trajectory is quite sensitive to the particular combination of m and n, and that these are additional parameters which help determine who wins or loses. (In the special case when $m = n$, a square

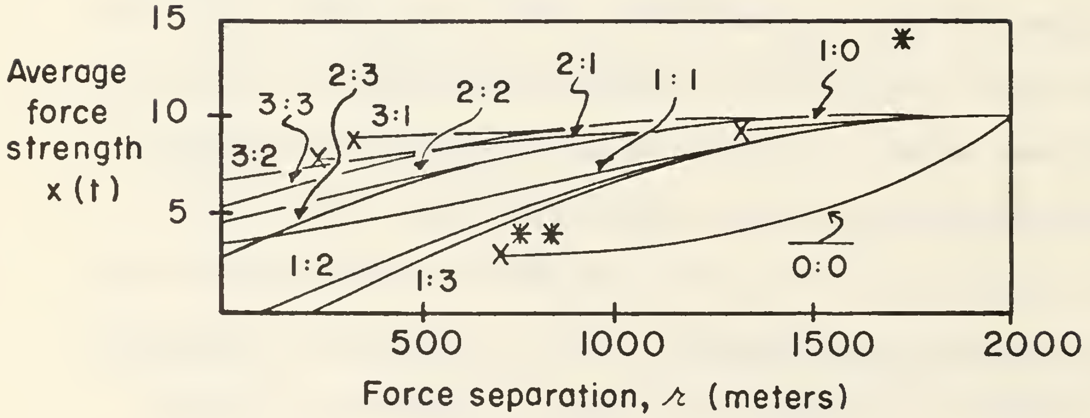
$R_o = R_\alpha = R_\beta = 2000$ meters

$\alpha_o = 0.06$ X-cas./(min. x Y-unit)

$\beta_o = 0.6$ Y-cas./(min. x X-unit)

$v = 5$ m. p. h.

$x_o = 10$, $y_o = 30$



* 1:0 denotes m = 1, n = 0.

** The symbol x denotes the end of a force-level trajectory due to the annihilation of enemy forces.

Figure 3. Force-level trajectories of X for various combinations of exponents, m and n, in power attrition-rate coefficients.

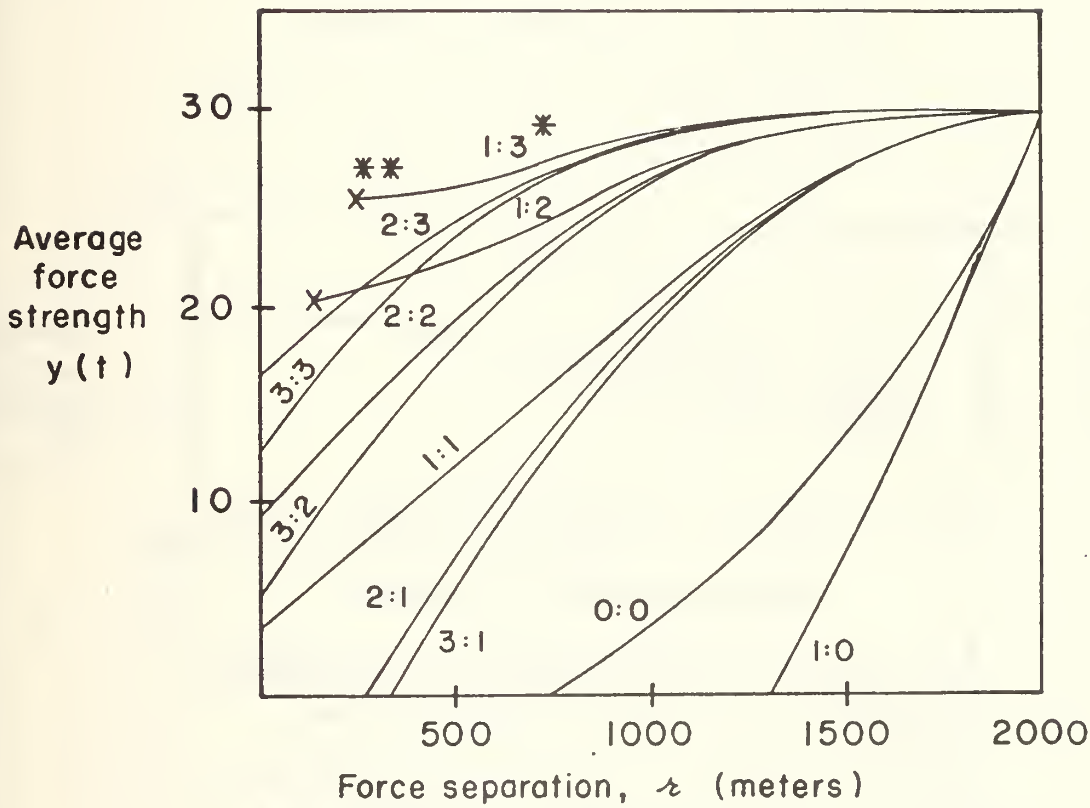
$R_o = R_\alpha = R_\beta$ = 2000 meters

α_o = 0.06 X-cas./(min. x Y-unit)

β_o = 0.6 Y-cas./(min. x X-unit)

v = 5 m.p.h.

x_o = 10, y_o = 30



* 1:3 denotes m = 1, n = 3.

** The symbol x denotes the end of a force-level trajectory due to the annihilation of enemy forces.

Figure 4. Force-level trajectories of Y for various combinations of exponents, m and n, in power attrition-rate coefficients.

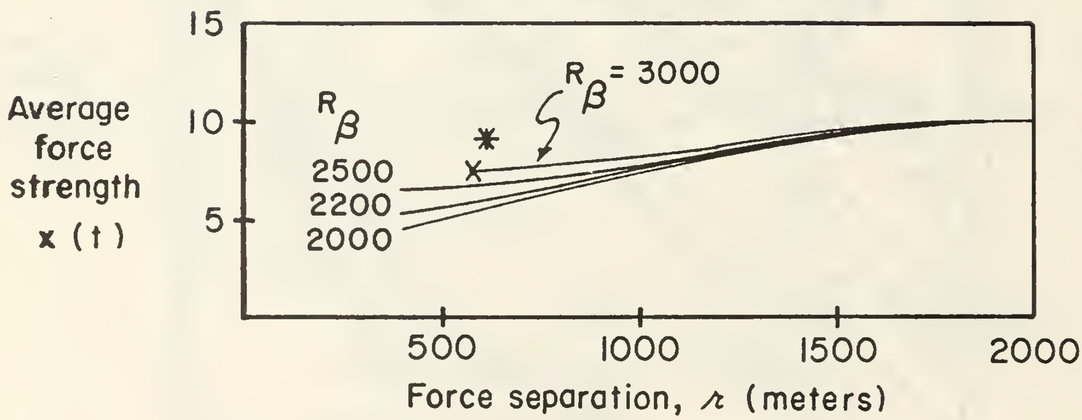
$R_o = R_\alpha = 2000$ meters

$\alpha_o = 0.06$ X-cas./(min. x Y-unit)

$\beta_o = 0.6$ Y-cas./(min. x X-unit)

$v = 5$ m.p.h.

$x_o = 10$, $y_o = 30$



*The symbol x denotes the end of a force-level trajectory due to the annihilation of enemy forces.

Figure 5. Force-level trajectories of X for various effective ranges of X-force weapons, R_β.

$R_0 = R_\alpha = 2000$ meters

$\alpha_0 = 0.06$ X-cas./(min. x Y-unit)

$\beta_0 = 0.6$ Y-cas./(min. x X-unit)

$v = 5$ m. p. h.

$x_0 = 10$, $y_0 = 30$

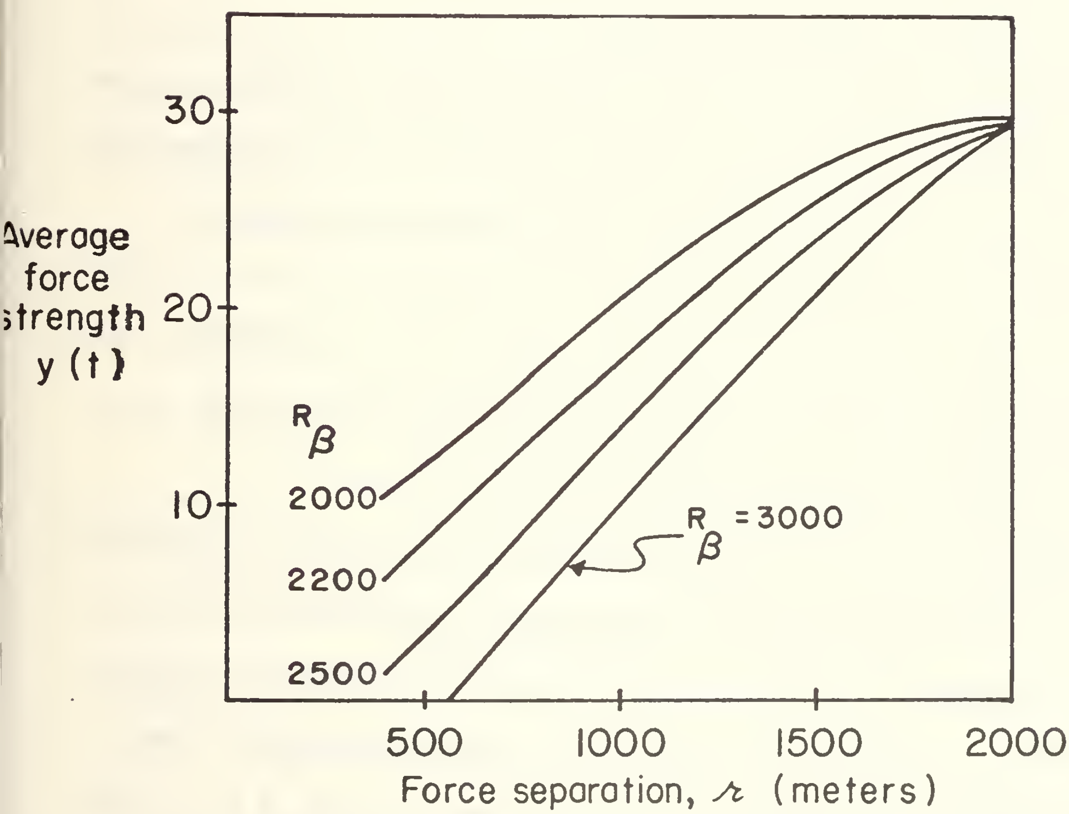


Figure 6. Force-level trajectories of Y for various effective ranges of X-force weapons, R_β.

law which is independent of these parameters relates the X- and Y-force strengths.) If we refer back to Figure 2, it should be clear why this is so: m and n reflect how weapon system kill rates increase with closing range to their peak values at "point blank" range. If we consider the Y-force to be the attacker who employs a weapon whose capability varies linearly with range, i.e., $m = 1$, then the battle may have quite different outcomes depending on the value of n. The reader should contrast the battle trajectories denoted as 1:0, 1:1, 1:2, and 1:3 in Figures 3 and 4.

We also see that we can use the initial trend of battle to forecast battle overcome only when we know the nature of the dependence of both weapon system capabilities upon range. Again, reference to Figures 3 and 4 should make this clear. For example, compare the outcomes for curves denoted as 1:2, 2:2, and 3:2. We also should note the "compounding" effect over time: a small advantage in range capability may materially effect battle outcome.

In Figures 5 and 6 we show the effect of increasing the effective range of the defender's weapons. Again we may consider the X-force to be the defender. For these computations we have held the opening range constant at $R_0 = R_\alpha = 2000$ meters. (It is a straightforward matter to extend this analysis to the case when $R_\alpha < R_0 \leq R_\beta$.) Both attrition-rate coefficients vary linearly with range, α_0 and β_0 have been held constant, and R_β has been varied. Except for the case in which $R_\beta = 2000$ meters when the solution to (22) is given by (7),

we know of no other instance in which there is a relationship between $x(t)$ and $y(t)$ that is independent of time (such as the classical Lanchester square law). The curves in Figures 5 and 6 exhibit the (obvious) qualitative result of increasing the long range capability of the defender's weapon system: more attacker casualties occur earlier in the battle, and these are then magnified by the "compounding nature" of the Lanchester-type equations (22). However, by considering such an attack scenario and using analysis such as the above, one might quantitatively assess the value of the long-range firepower capability of a weapon system (here the defender's weapons), giving consideration to the (hypothesized) dynamics of combat.

9. Comments.

In this section we discuss the significance, applications, and extensions of the results of this paper. Our research is complementary in nature to that on the Lanchester attrition-rate coefficient [2], [3], [6], [22], [30], [31]. To make use of our work one needs to know the range dependence of weapon system kill rates for the scenario under study. The demonstrated sensitivity of combat outcome to the type of range variation of weapon system kill rates makes important the determination of such range dependencies.

Our results extend the analytic capabilities for studying combat dynamics in those instances where time or range dependence is significant. Previous analytic results [3], [33] were limited to cases of weapon system performances described by kill rates with (1) the same

type of range dependence and (2) the same effective range. Our results allow both these restrictions to be removed. We may also view previous results as applying to the instance when the kill rate of one weapon system strictly dominates that of the enemy's (consider the plots of $a(t) = k_a t^m$ and $b(t) = k_b t^n$ when $m = n$). We have presented analysis results which apply when one weapon system does not dominate another at all ranges. Such a situation is shown in Figure 7.

We should also note that previous analytic results were developed only for when a square law related opposing force average strengths. In this instance one could view the effect of range dependent attrition rates as transforming the time (range) scale of the constant coefficient square law attrition process [33]. Our more general results apply even in those cases when (to the best of our knowledge) there is no time-dependent relationship between opposing force strengths.

Our results may be used to study the effects of mobility and weapon system range capabilities on combat dynamics. One can use our results to extend S. Bonder's analysis [5] of the implications and mobility considerations of Lanchester combat between forces employing weapon systems with appreciable range variations of kill rates. As noted above our results apply under less restrictive circumstances: opposing weapon systems' kill rates don't have to vary with range in the same manner, and both weapon systems don't necessarily have to have the same effective range.

Other extensions using the general solution (10) are possible for weapon systems that have different effective ranges. In this paper we have considered the illustrative case in which both weapon systems have

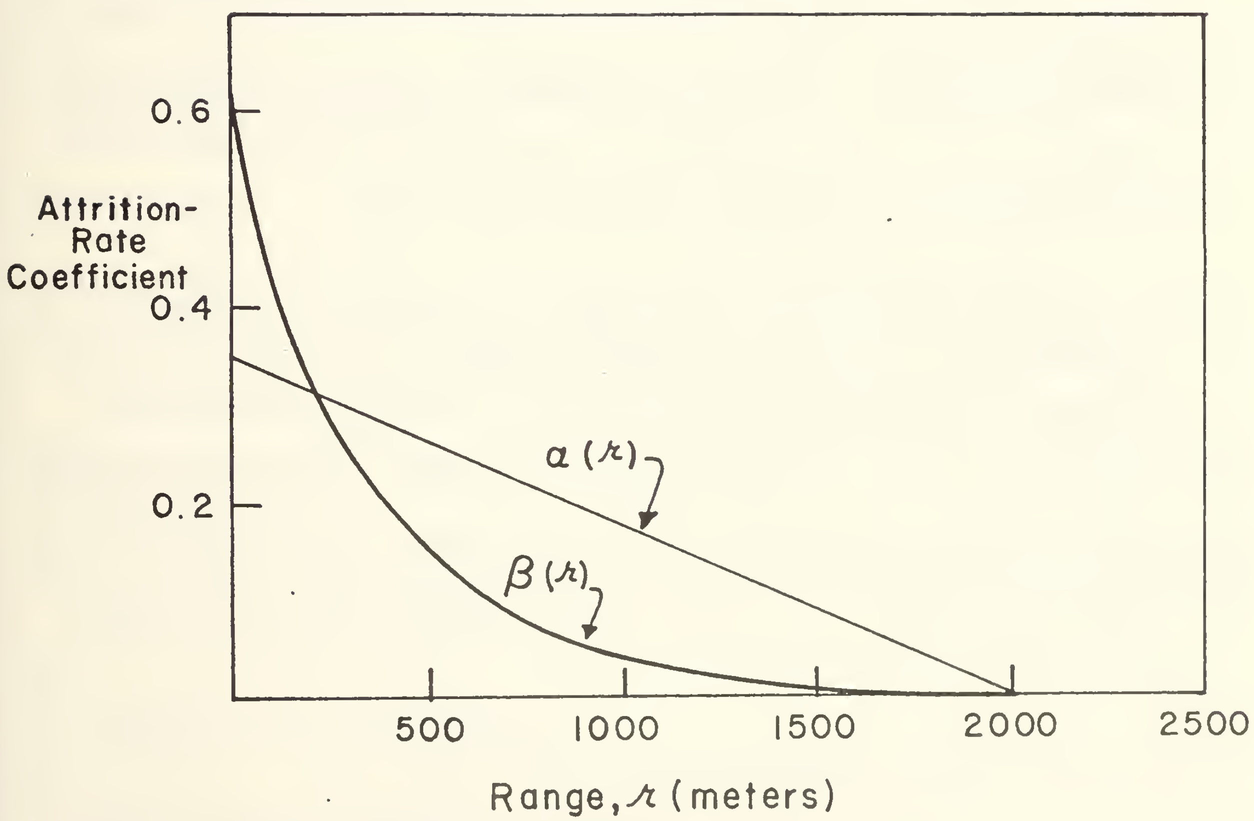


Figure 7. Example of weapon system not dominated at all ranges.

kill rates which very linearly with range. Extension to other range dependence combinations of attrition-rate coefficients is straightforward (but messy).

10. Summary.

We have developed a general solution to variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces. This general result was applied to two specific types of attrition-rate coefficients to yield new analytic results for Lanchester combat. In general, series solutions are obtained by our approach, but we have discussed the circumstances under which these may be expressed in terms of tabulated functions.

Solutions now exist for a wide class of variable-coefficient Lanchester-type equations of modern warfare as a result of this paper. We have extended analysis capabilities to include the following important cases: (1) opposing weapon systems whose kill rates have different range dependencies and (2) weapon systems with different effective ranges.

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Appendix I. Optimal Control of the Lanchester Stochastic Process.

1. Introduction.

In this appendix we present preliminary results from our study of the optimal control of the Lanchester stochastic process. Our goal is to determine whether the structure of the optimal allocation policy for the selection of target type at which to fire is affected by whether the attrition process is modelled as being deterministic or stochastic. Except for the pioneering work of Isbell and Marlow [27] (who optimized a truncated stochastic process and thus did not account for the true dynamics of combat), we are not aware of any other similar work on target selection in the Lanchester theory of combat when attrition is a stochastic process (although, of course, numerous papers have appeared (see, for example [12], [33]) concerning the application of optimal stochastic control theory to various other different tactical allocation problems). In our initial investigation here we shall consider a one-sided optimization problem, with our ultimate goal being the study of the corresponding stochastic game (i.e. two-sided decision problem).

It should be recalled that two basic approaches to analytic war gaming are (1) the Lanchester-type deterministic differential equation models [5], [6], [16] or (2) stochastic models (such as DYNTACS [3], a high-resolution simulation). In our past ONR sponsored research we have extensively studied the optimal control of the deterministic Lanchester attrition process [38], [39], [40], [41],

[42], [43]. In such optimization problems the differential equation models of combat are deterministic, always yielding the same result for given initial conditions. However, such attrition equations are commonly assumed to represent the mean course of battle (implying an underlying probability distribution [37]). In fact, one frequently refers to $x(t)$ as the average X-force level (this is well-known to be only an approximation [37]) and the model as an expected-value model.

As regards the optimal selection of the target type at which to fire, it is not clear whether such expected-value methodology would be appropriate to a situation with small numbers of combatants and a stochastic attrition mechanism (such as occurs with DYNTACS [3]). We base this opinion upon our consideration of the work of G. Clark, who studied (see Chapter 11 of [3] and [9]) differences between deterministic and stochastic homogeneous-force Lanchester formulations. He concludes that (see p. 11-19 of [3]) "the deterministic model would have difficulty approximating a stochastic simulation" with respect to the time history of force levels. Based on our consideration of the above results of G. Clark, we felt that it would be appropriate to investigate whether or not there are significant differences between optimal allocation policies for deterministic and stochastic attrition processes. If such a difference were to be observed, then this would raise the more basic question of whether or not different study results could be obtained merely by the choice of modelling technique, i.e. Lanchester-type expected value equations or Monte

Carlo simulation.

Our research approach is to consider the same scenario (prescribed duration battle between a homogeneous Y-force and a heterogeneous enemy, X_1 and X_2) and then compare the solution of the deterministic model with that of the corresponding stochastic one. To this end, we have developed a complete solution (in the sense that numberical values may be readily obtained by means of a digital computer program) to the two-versus-one prescribed duration battle (see Appendix G). In this appendix we consider the corresponding stochastic control problem. We will develop the basic equations from which the optimal target selection policy may be determined. When these equations are solved, results may be compared with those for the corresponding deterministic process. A M.S. thesis student (R. Powers, LT USN) is currently carrying out the details of the comparison of solutions. We hope to make a detailed comparison in our future ONR research.

It seems appropriate to briefly sketch some background on the theory of stochastic optimal control, which we will apply to our target selection problem. Bellman's principle of optimality (see p. 83 of [1]) and the concept of an optimal expected-value function (depending only upon the current state of the system) may be thought of as forming the basis for optimal stochastic control. (As pointed out by Wendell Fleming [20] it is a matter of taste how broadly one interprets the phrase optimal stochastic control; we accordingly avoid any definition here.) As pertains the problem at hand, it may be considered to be a special case of a general model discussed

by Kushner [30] (who cites similar work in England by Florentine [21]). S. Dreyfus has discussed the differences between some types of optimal control of stochastic systems [13] (see also [14]) (especially the difference between open-loop and closed-loop control, which are well-known to be equivalent for deterministic systems [13], [25]).

Recently, H. Kushner has written an introductory text on stochastic control [31], which contains many references to the literature (to which he has been a prolific contributor). Other survey papers (the first of which contains an extensive bibliography) are by W. Fleming [20] and P. Whittle [44].

2. The Lanchester Stochastic Process.

In 1914 in the British journal Engineering F. W. Lanchester [32] postulated that combat under the conditions of "modern warfare" between two homogeneous forces could be described by the equations

$$\frac{dx}{dt} = - ay,$$

$$\frac{dy}{dt} = - bx, \tag{1}$$

where a, b are commonly referred to as the Lanchester attrition-rate coefficients and x(t), y(t) are force levels. During World War II, B. Koopman suggested a reformulation of such a model in stochastic form [34]. Subsequent work has been by R. Snow [37], R. Brown [7], [8], and G. Clark [9]. B. Koopman has called the corresponding

stochastic process, the Lanchester stochastic process [29].

Before considering the control problem, it seems appropriate for us to review a few results for the Lanchester stochastic process. Consider combat between a homogeneous X-force and a homogeneous Y-force. We consider a stationary, Markov process (see any standard text [2], [11], [35] for a further discussion). Let the state probability be denoted by $P(m,n,t)$, where m is the number of x combatants alive and n is the number of Y combatants alive at time t. Thus,

$$P(m,n,t) = \text{Prob}\begin{bmatrix}\text{there are } m \text{ X and } n \text{ Y survivors}\\ \text{at time } t \text{ after battle begins}\end{bmatrix}.$$

Making standard assumptions (see [4]), we find that the state probability satisfies the following system of differential-difference equations

for $1 \leq m \leq M$ and $1 \leq n \leq N$ (also, we adopt the convention that $P(m,n,t) = 0$ for $m > M$ or $n > N$)

$$\frac{dP}{dt}(m,n,t) = P(m+1,n,t)\ A(m+1,n) + P(m,n+1,t)\ B(m,n=1)$$
$$- \{A(m,n) + B(m,n)\}\ P(m,n,t), \qquad (2)$$

where

$M(N)$ is the number of $X(Y)$ combatants at the beginning of battle at $t = 0$,

$A(m,n)$ is the rate of attrition of X forces; $A(0,n) = 0$,

and $B(m,n)$ is the rate of attrition of Y forces; $B(m,0) = 0$,

i.e.

$$\text{Prob}\begin{bmatrix}\text{one X casualty in interval} \\ \text{from } t \text{ to } t+\Delta t\end{bmatrix} = A(m,n)\Delta t.$$

(Moreover, we see that $P(m,n,t)$ is, more precisely, the transition probability

$$P(m,n,t) = P(m,n,t;M,N,t=0)$$
$$= \text{Prob}\begin{bmatrix}m \text{ X's and} & M \text{ X's and} \\ n \text{ Y's at } t & N \text{ Y's at } t=0\end{bmatrix}.)$$

Of course, the state space is discrete, i.e. $m = 0,1\ldots,M$ and $n = 0,1\ldots,N$. At the boundary of the system, i.e. $m = 0$ or $n = 0$, equation (2) takes the form

$$\frac{dP}{dt}(m,0,t) = P(m+1,0,t)A(m+1,0) + P(m,1,t)B(m,1) - P(m,0,t)A(m,0),$$

$$\frac{dP}{dt}(0,n,t) = P(0,n+1,t)B(0,n+1) + P(1,n,t)A(1,n) - P(0,n,t)B(0,n),$$

$$\frac{dP}{dt}(0,0,t) = P(1,0,t)A(1,0) + P(0,1,t)B(0,1), \tag{3}$$

while the initial conditions for (1) and (2) are taken to be

$$P(m,n,t=0) = \begin{cases}1 & \text{for } m=M, n=N, \\ 0 & \text{for } m\neq M, n\neq N.\end{cases} \tag{4}$$

Let us adopt the following terminology for the attrition-rate coefficients (and hence the process itself). We shall say that we have a

(a) linear-law attrition process when

$$A(m,n) = amn,$$
$$B(m,n) = bmn, \tag{5}$$

and

(b) square-law attrition process when

$$A(m,n) = \beta m + an,$$
$$B(m,n) = bm + \alpha n, \tag{6}$$

where α, β may be referred to as operational loss rates.

Although it is well-known that (2) through (4) yield an exponential solution (the well-known Chapman-Kolmogorov equation expresses the semi-group property of the state probabilities (see [29])) when $A(m,n)$ and $B(m,n)$ are specified (for example, by (6)), general solutions have been optained to this system only in a few special cases. In the special case when $a + \alpha = b + \beta$, Isbell and Marlow [27] developed a general solution to (2) through (4) for a square-law stochastic attrition process. Recently, Clark (see pp. 102-104 of [9]) obtained the general solution to the linear-law stochastic attrition process (i.e. $A(m,n)$ and $B(m,n)$ are given by (5)). One reason why we have reviewed this material is to now point out to the reader that a general solution to (2) through (4) only exists for a linear-law attrition process and is very complex (see pp. 102-104 of [9]). In considering the optimal control of the Lanchester stochastic (square-law) process, we will encounter a

similar system of equations for the optimal expected-value function. Keeping in mind that a general solution to the corresponding equations (2) through (4) for the state probabilities of the square-law stochastic attrition process, we have not (seriously) tried to develop a general solution for those equations satisfied by the optimal expected-value function in view of such difficulties.

Additionally, using the above noted solutions for the Lanchester stochastic process, Clark (following results in [34] and qualitative results in [37]) made a comparison of the average force levels in the stochastic attrition process (denoted as $\bar{m}(t)$ and $\bar{n}(t)$) and the corresponding force levels $x(t)$ and $y(t)$ in the deterministic model (such as (1)). Unlike the corresponding situation for the Yule-Ferry linear birth process (see pp. 77-78 of [2] or pp. 156-159 of [11]), there is a bias (due to "boundary effects") in dynamical behavior of $x(t)$ and $y(t)$ as compared with $\bar{m}(t)$ and $\bar{n}(t)$ for the same a's and b's. This is a major result of Clark's careful investigation in which several numerical examples are given to prove such points. Clark's solution to the stochastic linear-law process was important in making such a comparison. This fact that the average of the Lanchester stochastic process does not behave identically to the corresponding force levels $x(t)$ and $y(t)$ computed according to the corresponding deterministic model has led us to make the present investigation.

3. The Optimal Control Problem.

In Appendix G we studied the following optimal control problem for the deterministic Lanchester attrition process:

$$\underset{\phi(t)}{\text{maximize}} \{ry(T)-px_1(T)-qx_2(T)\} \text{ with } T = T_1 \text{ specified,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1-b_2x_2, \tag{7}$$

$$x_1, x_2, y \geq 0 \qquad \text{and} \qquad 0 \leq \phi \leq 1,$$

where

$x_1(t)$, $x_2(t)$, $y(t)$ are force levels,

p, q, r are utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

and ϕ is the fraction of Y fire directed at X_1.

The initial conditions for the above problem are

$$x_1(t=0) = x_1^o,$$
$$x_2(t=0) = x_2^o,$$
$$y(t=0) = y_0.$$

The above combat scenario is for a prescribed (fixed) length of time T_1 (although one must, of course, give consideration to "premature" terminations as we did in Appendix G). For convenience, we have chosen to call this problem the "prescribed duration battle."

Let us take the following for our optimal control problem in which the combat is described as a stationary Markov process (with a discrete state space):

$$\underset{\phi(t)}{\text{maximize}} \{rn(T)-pm_1(T)-qm_2(T)\} \text{ with } T = T_1 \text{ specified,}$$

subject to: casualties occur randomly as a stationary Markov process corresponding to the deterministic process (7), (8)

$$m_1, m_2, n \geq 0 \qquad \text{and} \qquad 0 \leq \phi \leq 1,$$

where

$m_1(t)$, $m_2(t)$, $n(t)$ are force levels (integers),

p, q, r are again utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

and ϕ is the fraction of Y fire directed at X_1.

Furthermore, $\phi(t)$ is restricted to take on values from the set $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$.

4. <u>Development of Fundamental Functional Equation</u>.

Let $S(\tau,m_1,m_2,n)$ denote the optimal expected-value function

(see [14]). Then

$$S(\tau,m_1,m_2,n) = S(\tau,m_1(\tau),m_2(\tau),n(\tau)),$$

and

$$S(\tau,m_1,m_2,n) = \underset{\phi(\tau)\varepsilon\Phi}{\text{maximum}}\ E_{\vec{m},\tau}\left[rn(\tau=0)-pm_1(\tau=0)-qm_2(\tau=0)\right], \quad (9)$$

where

the system is in state m_1, m_2, n at time τ (i.e. there are m_1 of the X_1-forces, etc.),

the system is in state $m_1(\tau=0)$, $m_2(\tau=0)$, $n(\tau=0)$ at $\tau = 0$,

Φ is the class of admissible controls (i.e. $\phi(\tau)$ must always be chosen from the set of rational numbers $\{0,\frac{1}{n(\tau)},\frac{2}{n(\tau)},\ldots,1\}$,

$\tau = T - t$ is the backwards time from the end of battle (which begins at $t=0$),

$E_{\vec{m},\tau}$ denotes mathematical expectation (with respect to all possible states at $\tau=0$) given that $\vec{m}(\tau) = (m_1(\tau),m_2(\tau),n(\tau))$,

casualties occur in a random fashion between t and T.

In other words, $S(\tau,m_1,m_2,n)$ is the maximum return that we get on the average when we start with force levels m_1,m_2, and n at $t = T-\tau$, follow an optimal policy $\phi^*(s)$ (chosen from the class of admissible policies Φ) for $t \le s \le T$, and casualties occur in a

random fashion.

We consider that casualties occur as a stationary Markov process with discrete state space (or discontinuous Markov process). Specifically, we assume that

(1) the attrition process is a stationary Markov process corresponding to a deterministic Lanchester square-law attrition process; this is equivalent to assuming

(a) the future occurrences of casualties depend only on the state of the system at t and not on past history,

(b) the transition probabilities depend on only the state of the system,

(c)

$$\text{Prob}\begin{bmatrix}\text{one } X_1 \text{ casualty} \\ \text{in interval } \Delta t\end{bmatrix} = \phi a_1 m\Delta t,$$

$$\text{Prob}\begin{bmatrix}\text{one } X_2 \text{ casualty} \\ \text{in interval } \Delta t\end{bmatrix} = (1-\phi)a_2 n\Delta t,$$

$$\text{Prob}\begin{bmatrix}\text{one } Y \text{ casualty} \\ \text{in interval } \Delta t\end{bmatrix} = (b_1 m_1 + b_2 m_2)\Delta t,$$

where $\phi a_1 n$ is X_1 casualty rate, etc.,

(d) $$\text{Prob}\begin{bmatrix}\text{more than one casualty} \\ \text{in interval } \Delta t\end{bmatrix} = 0((\Delta t)^2),$$

where following Copson (see p. 35 of [10]) $0(x)$ denotes dependence on x such that

$$\lim_{x \to 0} \frac{0(x)}{x} = \underline{\text{a constant}},$$

(2) the Y-forces have perfect information as to the state of the system at t and the expected casualty rates,

(3) the Y-forces can instantaneously shift fire from any target at any time,

(4) the length of the battle is known.

Further defining our notation, we have

$m_1(t), m_2(t), n(t)$ are number of X_1-forces, X_2-forces, Y-forces, respectively, as a function of time t(restricted to be non-negative integer),

and

$\phi(t)$ is fraction of Y-force total fire directed at X_1 (restricted to be rational between 0 and 1 with denominator $n(t)$).

Thus, we have

<u>state variables</u>: $m_1(t), m_2(t), n(t)$,

<u>decision (or control) variable</u>: $\phi(t)$,

where

$$\phi(t) \in \Phi = \left\{0, \frac{1}{n(t)}, \frac{2}{n(t)}, \ldots, \frac{n(t)-1}{n(t)}, 1\right\}.$$

To be more precise $\phi = \phi\,(m_1, m_2, n, t)$ is a closed-loop (or feedback) control (see Appendix F for a further discussion and contrast with open-

loop control), although for notational convenience we don't explicitly show this dependence below. Furthermore, let τ denote the "backwards time" from the end of battle, i.e. $\tau = T-t$.

To develop the <u>fundamental functional equation</u> for the optimal expected-value function, we begin by considering any interval of "backwards time" of length $\Delta\tau$ which occurs from $\tau-\Delta\tau$ to τ. There are five exhaustive and mutually exclusive possibilities for random events to occur in such an interval. These are

(1) one X_1 casualty occurs,

(2) one X_2 casualty occurs,

(3) one Y casualty occurs,

(4) no casualty occurs,

(5) more than one casualty occurs.

Let us now examine each of these cases and develop expected returns.

(1) <u>One X_1 casualty occurs in $\Delta\tau$</u>:

By our assumptions above, we have for the probability of occurrence of this event

$$\text{Prob[one } X_1 \text{ casualty occurs in } \Delta\tau] = \phi a_1 n\Delta\tau. \tag{10}$$

Given that one X_1 casualty is realized in the interval from τ to $\tau - \Delta\tau$, the optimal target selection policy for Y will consider the maximum expected value for the return functional as casualties continue to occur randomly from $\tau - \Delta\tau$ to $\tau = 0$. This maximum expected value is $S(\tau-\Delta\tau, m_1(\tau-\Delta\tau), m_2(\tau-\Delta\tau), n(\tau-\Delta\tau))$ where $m_1(\tau-\Delta\tau) = m_1(\tau)-1$, $m_2(\tau-\Delta\tau) = m_2(\tau)$, and $n(\tau-\Delta\tau) = n(\tau)$.

(2) <u>One X_2 casualty occurs in $\Delta\tau$</u>:

Similarly, we have that

$$\text{Prob[one } X_2 \text{ casualty occurs in } \Delta\tau] = (1-\phi)a_2 n\, \Delta\tau, \tag{11}$$

with the optimal expected-value function $S(\tau-\Delta\tau, m_1(\tau), m_2(\tau)-1, n(\tau))$.

(3) <u>One Y casualty occurs in $\Delta\tau$</u>:

Similarly, we have that

$$\text{Prob[one } Y \text{ casualty occurs in } \Delta\tau] = (b_1 m_1 + b_2 m_2)\, \Delta\tau, \tag{12}$$

with the optimal expected-value function $S(\tau-\Delta\tau, m_1(\tau), m_2(\tau), n(\tau)-1)$.

(4) <u>No casualty occurs in $\Delta\tau$</u>:

$$\text{Prob[no casualty occurs in } \Delta\tau] =$$

$$1 - \{\phi a_1 n + (1-\phi) a_2 n + b_1 m_1 + b_2 m_2\}\, \Delta\tau + O((\Delta\tau)^2). \tag{13}$$

The optimal expected-value function at $\tau-\Delta\tau$ is then accordingly $S(\tau-\Delta\tau, m_1(\tau), m_2(\tau), n(\tau))$.

(5) <u>More than one casualty occurs in $\Delta\tau$</u>:

The probability that this event occurs is $O((\Delta\tau)^2)$.

Now, by the standard dynamic programming argument which combines the probabilities of events (1) through (5) above with the maximum expected return to be achievable given these events occur, we obtain the expression

$$S(\tau,m_1,m_2,n) = \underset{\substack{0\le\phi\le 1\\ \phi\varepsilon\,\Phi}}{\text{maximum}}\{[1-\Delta\tau\{\phi a_1 n+(1-\phi)a_2 n +$$

$$b_1 m_1+b_2 m_2\}]S(\tau-\Delta\tau,m_1,m_2,n) + \phi a_1 n\ \Delta\tau\ S(\tau-\Delta\tau,m_1-1,m_2,n)$$

$$+ (1-\phi)a_2 n\ \Delta\tau\ S(\tau-\Delta\tau,m_1,m_2-1,n) +$$

$$(b_1 m_1+b_2 m_2)\ \Delta\tau\ S(\tau-\Delta\tau,m_1,m_2,n-1)\}. \qquad (14)$$

Rearranging terms in (14) and taking the limit as $\Delta\tau\to 0$, we obtain the fundamental functional equation for the optimal expected-value function $S(\tau,m_1,m_2,n)$

for $m_1 > 0$, $m_2 > 0$, $n > 0$:

$$\frac{dS}{d\tau}(\tau,m_1,m_2,n) = (b_1 m_1+b_2 m_2)\{S(\tau,m_1,m_2,n-1) - S(\tau,m_1,m_2,n)\}$$

$$+ n \underset{\substack{0\le\phi\le 1\\ \phi\varepsilon\,\Phi}}{\text{maximum}} [\phi a_1 \{ S(\tau,m_1-1,m_2,n) - S(\tau,m_1,m_2,n)\}$$

$$+ (1-\phi)a_2\{S(\tau,m_1,m_2-1,n) - S(\tau,m_1,m_2,n)\}], \qquad (15)$$

with the boundary condition at $t = T$

$$S(\tau=0,m_1,m_2,n) = rn(\tau)-pm_1(\tau)-qm_2(\tau), \qquad (16)$$

where

$m_1, m_2,$ and n are integers,

and

$$\Phi = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\}.$$

The special forms of (15) in cases when $m_1=0$, etc. will be given later.

More concisely, we could have said that (15) results from combination of the well-known formalism of dynamic programming with the retrospective (backward) probabilistic evolution of the system over time (c.f. [21], [31]). It is to be noted that (15) is a special case of an equation given by Kushner in 1962 [30].

If we take (15) to be the fundamental (or basic) equation for $S(\tau, m_1, m_2, n)$, then (14) may be considered to be the simplest finite difference approximation to it, i.e. the result of applying the well-known Euler's method to (15) (see pp. 130-131 of [53]). (Of course, a method employing a higher order approximation scheme (see pp. 132-140 of [24]) may be necessary under many circumstances.) We will find this point of view convenient when we consider developing a solution to (15).

Finally, it seems appropriate for us to note that equations (15) have precisely the same structure as the equations for the Lanchester (square-law attrition, since the attrition rates have the same form as (6)) stochastic process for which (to the best of our knowledge) no general solution has been developed. This last fact shows that although it is a well-known and trivial theoretical result

for $m_1 = 0,\ m_2 = 0,\ n \geq 0$,

$$S(\tau,0,0,n) = nr, \tag{18}$$

for $m_1 = 0,\ m_2 > 0,\ n > 0$,

$$\frac{dS}{d\tau}(\tau,0,m_2,n) = b_2m_2\ \{S(\tau,0,m_2,n-1) - S(\tau,0,m_2,n)\}$$
$$+\ a_2n\{S(\tau,0,m_1-1,n) - S(\tau,0,m_2,n)\}, \tag{19}$$

for $m_1 > 0,\ m_2 = 0,\ n > 0$,

$$\frac{dS}{d\tau}(\tau,m_1,0,n) = b_1m_1\{S(\tau,m_1,0,n-1) - S(\tau,m_1,0,n)\}$$
$$+\ a_1n\{S(\tau,m_1-1,0,n) - S(\tau,m_1,0,n)\}. \tag{20}$$

Equations (15) through (20) are the complete system of equations for the optimal expected-value function in the optimal control of the Lanchester stochastic process.

For $m_1 > 0,\ m_2 > 0,\ n > 0$ the optimal target selection policy is determined by the maximization operation in (15), and hence

$$\phi^*(\tau) = \begin{cases} 1 & \text{for } W(\tau,m_1,m_2,n) > 0, \\ 0 & \text{for } W(\tau,m_1,m_2,n) < 0, \end{cases} \tag{21}$$

where we shall refer to $W(\tau,m_1,m_2,n)$ as the "switching function." It is defined by

for $m_1 > 0,\ m_2 > 0,\ n > 0$,

$$W(\tau,m_1,m_2,n) = a_1\{S(\tau,m_1-1,m_2,n) - S(\tau,m_1,m_2,n)\}$$
$$-\ a_2\{S(\tau,m_1,m_2-1,n) - S(\tau,m_1,m_2,n)\}. \tag{22}$$

Let us observe that at the end of the battle at $t = T$, we may combine (16), (21), and (22) to obtain

$$\phi^*(\tau=0) = \begin{cases} 1 & \text{for } a_1 p > a_2 q, \\ 0 & \text{for } a_1 p < a_2 q, \end{cases} \tag{23}$$

which the reader will recall as the same result previously obtained for the optimal control of the deterministic process (see Appendices A, F, and G).

As pointed out in Section 4, equations (15) through (20) have the same form as the equations for the Lanchester square-law attrition stochastic process (i.e. equations (2) through (4) when the attrition rates are given by (6)). (We note that (15) reduces to a first order system of ordinary differential equations for $S(\tau,m_1,m_2,n)$ when ϕ^* as determined by (21) is used. Solving for $S(\tau,m_1,m_2 n)$ for $m_1 = 0,1,2,\ldots,$etc., we can then determine ϕ^* by (21). The synthesis of optimal control by combination of the control law (21) with integration of a system of differential equations is similar to that for deterministic optimal control problems.) We have pointed out that (to the best of our knowledge) a general solution has not been obtained for the Lanchester stochastic process equations. Hence, it seems unlikely that an analytic solution is readily obtainable for (15) through (20) (although such a solution may be considered to be a matrix exponential (see above)).

Nevertheless, it is of value to develop a partial solution. For example, since we use finite difference methods to generate an approxi-

mate solution, it is desirable to check the adequacy of the approximation (in particular, the "time step size" used in the numerical propagation of the approximate solution by "marching ahead in time"). This is easily done by comparing the approximate solution, denoted by $\hat{S}$, to the exact analytic solution, denoted by S. Hence, a partial analytic solution is useful.

Careful consideration of (15) through (20) reveals that there are restrictions on the order in which the optimal expected-value functions $S(\tau,m_1,m_2,n)$ for $m_1=0,1,2,\ldots,$etc. can be computed. In particular, one admissible sequence for building up the solution through $S(\tau,1,1,1)$ is shown below in Table 1.

m_1	m_2	n
0	0	0
1	0	0
0	1	0
0	0	1
1	1	0
0	1	1
1	0	1
1	1	1

Table I. Admissible Order for Computing Optimal Expected-Value Functions (admissible order is from top to bottom).

Thus, we readily successively compute using (17) through (20)

$$S(\tau,0,0,0) = 0,$$

$$S(\tau,1,0,0) = -p,$$

$$S(\tau,0,1,0) = -q$$

$$S(\tau,0,0,1) = r,$$

$$S(\tau,1,1,0) = -p-q,$$

$$S(\tau,0,1,1) = \left(\frac{b_2r-a_2q}{a_2+b_2}\right) e^{-(a_2+b_2)\tau} + \left(\frac{a_2r-b_2q}{a_2+b_2}\right),$$

$$S(\tau,1,0,1) = \left(\frac{b_1r-a_1p}{a_1+b_1}\right) e^{-(a_1+b_1)\tau} + \left(\frac{a_1r-b_1p}{a_1+b_1}\right). \tag{24}$$

Furthermore, for $m_1 = 1$, $m_2 = 1$, $n = 1$, by (24) equations (15) and (16) become

$$\frac{dS}{d\tau}(\tau,1,1,1) = -(b_1+b_2)\{S(\tau,1,1,1) + (p+q)\}$$

$$+ \underset{\substack{0 \le \phi \le 1 \\ \phi = 0 \text{ or } 1}}{\text{maximum}} [\phi a_1\{S(\tau,0,1,1) - S(\tau,1,1,1)\} +$$

$$(1-\phi)\, a_2\{S(\tau,1,0,1) - S(\tau,1,1,1)\}],$$

$$\text{with } S(\tau=0,1,1,1) = r - p - q, \tag{25}$$

where $S(\tau,0,1,1)$ and $S(\tau,1,0,1)$ are given by (24). It is convenient to denote that "switching function" as $w(\tau)$

$$w(\tau) = W(\tau,1,1,1) =$$

$$a_1\{S(\tau,0,1,1) - S(\tau,1,1,1)\} - a_2\{S(\tau,1,0,1) - S(\tau,1,1,1)\}, \tag{26}$$

so that (21) becomes

$$\phi^*(\tau) = \begin{cases} 1 & \text{for} \quad w(\tau) > 0, \\ 0 & \text{for} \quad w(\tau) < 0. \end{cases} \tag{27}$$

We observe that

$$w(\tau{=}0) = a_1 p - a_2 q, \tag{28}$$

so that (23) continues to hold.

Using (23), (24), (26), and (27), we may readily solve (25). As in the deterministic case (see Appendix G), there are two cases that must be distinguished

$$\text{Case (1)} \quad a_1 p \geq a_2 q,$$

$$\text{Case (2)} \quad a_1 p < a_2 q.$$

For Case (1): $a_1 p \geq a_2 q$, we have that $\phi^*(\tau) = 1$ for $0 \leq \tau \leq \tau_1$, where τ_1 denotes the "backwards time" of the first switch in the optimal target selection policy. τ_1 is the smallest τ which satisfies $w(\tau{=}\tau_1) = 0$ with $w(\tau)$ given by (26).

for $0 \leq \tau \leq \tau_1$ when $a_1 p \geq a_2 q$ $(\phi^*(\tau){=}1)$

$$S(\tau,1,1,1) = \frac{a_1(b_2 r - a_2 q)}{(a_1+b_1-a_2)(a_2+b_2)} e^{-(a_2+b_2)\tau} +$$

$$\left\{\frac{[(b_1-a_2)(b_1+b_2)+a_1b_1]r}{(a_1+b_1-a_2)(a_1+b_1+b_2)} - \frac{a_1p}{(a_1+b_1+b_2)} + \frac{a_1a_2q}{(a_1+b_1-a_2)(a_1+b_1+b_2)}\right\}e^{-(a_1+b_1+b_2)\tau}$$

$$+\left\{\frac{a_1a_2r}{(a_2+b_2)(a+b_1+b_2)} - \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} - \frac{[(b_1+b_2)(a_2+b_2)+a_1b_2]q}{(a_2+b_2)(a_1+b_1+b_2)}\right\}. \quad (29)$$

We note that τ_1 might be equal to $+\infty$, i.e. we never switch. Assuming that a switch in targets does occur, however, let us denote $S(\tau=\tau_1,1,1,1)$ by S_0 where, as we recall, τ_1 is the smallest τ which satisfies $w(\tau=\tau_1) = 0$. Then, we have that $\phi^*(\tau) = 0$ for $\tau_1 < \tau \leq \tau_2$, where τ_2 denotes the "backwards time" of the second switch in the optimal target selection policy. Then, we have

for $\tau_1 < \tau \leq \tau_2$ when $a_1p \geq a_2q$ $(\phi^*(\tau) = 0)$

$$S(\tau,1,1,1) = \frac{a_2(b_1r-a_1p)}{(a_2+b_2-a_1)(a_1+b_1)}\left\{e^{-(a_1+b_1)\tau} - e^{(a_2+b_2)(\tau_1-\tau)-a_1\tau_1-b_1\tau}\right\}$$

$$+\left\{S_0 - \frac{a_1a_2r}{(a_1+b_1)(a_2+b_2+b_1)} + \frac{[(b_1+b_2)(a_1+b_1)+a_2b_1]p}{(a_1+b_1)(a_2+b_2+b_1)}\right.$$

$$\left.+\frac{(b_1+b_2)q}{(a_2+b_2+b_1)}\right\}e^{(a_2+b_2+b_1)(\tau_1-\tau)} + \left\{\frac{a_1a_2r}{(a_1+b_1)(a_2+b_2+b_1)}\right.$$

$$\left.-\frac{[(b_1+b_2)(a_1+b_1)+a_2b_1]p}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{(b_1+b_2)q}{(a_2+b_2+b_1)}\right\}. \quad (30)$$

Again, we note that τ_2 might be equal to $+\infty$, i.e. we might never switch targets a second time. Assuming that a second switch in targets does occur, we have $\phi^*(\tau) = 1$ for $\tau_2 < \tau \leq \tau_3$. We have not carried the computation of $S(\tau,1,1,1)$ past τ_2.

For Case (2): $a_1p < a_2q$, we have that $\phi^*(\tau) = 0$ for $0 \leq \tau \leq \tau_1$, where τ_1 denotes the "backwards time" of the first switch in the optimal target selection policy. τ_1 is the smallest τ which satisfies $w(\tau=\tau_1) = 0$ with $w(\tau)$ given by (26).

for $0 \leq \tau \leq \tau_1$ when $a_1p < a_2q$ $(\phi^*(\tau) = 0)$

$$S(\tau,1,1,1) = \frac{a_2(b_1r-a_1p)}{(a_2+b_2-a_1)(a_1+b_1)} e^{-(a_1+b_1)\tau} +$$

$$\left\{\frac{[(b_2-a_1)(b_1+b_2)+a_2b_2]r}{(a_2+b_2-a_1)(a_2+b_2+b_1)} + \frac{a_1a_2p}{(a_2+b_2-a_1)(a_2+b_2+b_1)} - \frac{a_2q}{(a_2+b_2+b_1)}\right\} e^{-(a_2+b_2+b_1)\tau}$$

$$+ \left\{\frac{a_1a_2r}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{[(b_1+b_2)(a_1+b_1)+a_2b_1]p}{(a_1+b_1)(a_2+b_2+b_1)} - \frac{(b_1+b_2)q}{(a_2+b_2+b_1)}\right\}. \quad (31)$$

We note that τ_1 might be equal to $+\infty$, i.e. we never switch. Assuming that a switch in targets does occur, however, let us denote $S(\tau=\tau_1,1,1,1)$ by S_0 where, as we recall, τ_1 is the smallest τ which satisfies $w(\tau=\tau_1) = 0$. Then, we have that $\phi^*(\tau) = 1$ for $\tau_1 < \tau \leq \tau_2$, where τ_2 denotes the "backwards time" of the second switch in the optimal target selection policy. Then, we have

for $\tau_1 < \tau \leq \tau_2$ when $a_1 p > a_2 q$ $(\phi^*(\tau) = 1)$

$$S(\tau,1,1,1) = \frac{a_1(b_2 r - a_2 q)}{(a_1+b_1-a_2)(a_2+b_2)} \left\{ e^{-(a_2+b_2)\tau} - e^{(a_1+b_1)(\tau_1-\tau)-a_2\tau_1-b_2\tau} \right\}$$

$$+ \left\{ S_0 - \frac{a_1 a_2 r}{(a_1+b_1+b_2)(a_2+b_2)} + \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} + \right.$$

$$\left. \frac{[(b_1+b_2)(a_2+b_2)+a_1 b_2]q}{(a_1+b_1+b_2)(a_2+b_2)} \right\} e^{(a_1+b_1+b_2)(\tau_1-\tau)}$$

$$+ \left\{ \frac{a_1 a_2 r}{(a_1+b_1+b_2)(a_2+b_2)} - \frac{(b_1+b_2)p}{(a_1+b_1+b_2)} - \frac{[(b_1+b_2)(a_2+b_2)+a_1 b_2]q}{(a_1+b_1+b_2)(a_2+b_2)} \right\}. \quad (32)$$

Again, we note that τ_2 might be equal to $+\infty$, i.e. we might never switch targets a second time. Assuming that a second switch in targets does occur, we have $\phi^*(\tau) = 0$ for $\tau_2 < \tau \leq \tau_3$. We have not carried the computation of $S(\tau,1,1,1)$ past τ_2.

Although the above constitutes a complete solution for $S(\tau,1,1,1)$ and hence $\phi^*(\tau)$ via $w(\tau)$, these results are complex enough that it is not immediately clear how $\phi^*(\tau)$ changes over time and/or depends on the variable model parameters. (We recall that in the deterministic case when $x_1(T) > 0$ and $x_2(T) > 0$, the conditions $a_1 p \geq a_2 q$ and $a_1 b_1 > a_2 b_2$ implied that $\phi^*(\tau) = 1$ for the entire battle.) We plan to analyze such aspects more thoroughly

both analytically and computationally in the future.

6. Development of Numerical Solution.

As we have already discussed several times above, a general solution for $S(\tau, m_1, m_2, n)$ which satisfies (15) through (20) for $m_1 \geq 0$, $m_2 \geq 0$, $n \geq 0$ has not been obtained and prospects do not seem bright for this. With the advent of modern high-speed digital computers, finite difference methods of obtaining an approximate solution are commonly used under such circumstances.

Euler's method (see pp. 130-131 of [24]) yields the simplest finite difference approximation for (15). Let us denote the approximation to the optimal expected-value function by $\hat{S}$. We shall compute values for the approximation to the optimal expected-value function at discrete points in time separated by a constant amount $\Delta\tau$. We let $\tau = \ell\Delta\tau$ so that $T = L\Delta\tau$. Then (15) may be approximated by

for $m_1 > 0$, $m_2 > 0$, $n > 0$:

$$S((\ell+1)\Delta\tau, m_1, m_2, n) = \{1-(\Delta\tau)(b_1 m_1 + b_2 m_2)\} S(\ell\Delta\tau, m_1, m_2, n)$$

$$+ (\Delta\tau)(b_1 m_1 + b_2 m_2) S(\ell\Delta\tau, m_1, m_2, n-1)$$

$$+ y(\Delta\tau) \underset{\substack{0 \leq \phi \leq 1 \\ \phi \in \Phi}}{\text{maximum}} [\phi a_1\{S(\ell\Delta\tau, m_1 - 1, m_2, n)$$

$$- S(\ell\Delta\tau, m_1, m_2, n)\} + (1-\phi) a_2\{S(\ell\Delta\tau, m_1, m_2 - 1, n) - S(\ell\Delta\tau, m_1, m_2, n)\}], \quad (33)$$

for $\ell = 0,1,\ldots,L-1$ with boundary condition

$$S(0,m_1,m_2,n) = rn - pm_1 - qm_2,$$

where

$$m_1,\ m_2,\ \text{and}\ n\ \text{are integers}$$

and

$$\Phi = \left\{0,\frac{1}{n},\frac{2}{n},\ .\ .\ .,\frac{(n-1)}{n},1\right\}.$$

We earlier had seen how (33) arose in the derivation of (15). Similar approximations may be developed for (19) and (20).

As we noted above, consideration of (15) through (20) yields that there are restrictions on the order in which the optimal expected-value function S (or its approximation $\hat{S}$). We observe that the computation $\hat{S}((\ell+1)\Delta\tau,m_1,m_2,n)$ depends upon

$$\hat{S}(\ell\Delta\tau,m_1,m_2n),$$
$$\hat{S}(\ell\Delta\tau,m_1-1,m_2,n),$$
$$\hat{S}(\ell\Delta\tau,m_1,m_2-1,n),$$
$$\hat{S}(\ell\Delta\tau,m_1,m_2,n-1).$$

The dependence upon the state variables is shown in Figure 1 below.

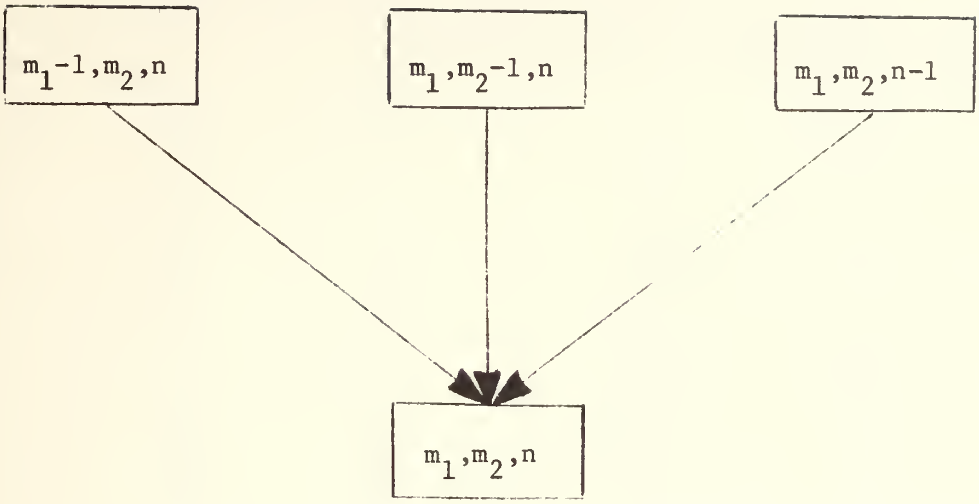

Figure 1. Dependence of Optimal Expected-Value Function on Discrete State Variables.

Based on the dependence shown in Figure 1, the solution can be "built-up" as shown in Table II. the general case is shown in Table III.

It remains to discuss the adequacy of the finite difference approximation (33). It is well-known (see pp. 130-145 in [24]) that Euler's method yields a finite difference approximation for such a system of differential equations that is both consistent and stable so that the approximate solution $\hat{S}$ can be guaranteed to converge to the exact analytic solution S as $\Delta\tau \to 0$ (and $L \to \infty$) [36]. However, $\Delta\tau$ must not be too large in order to keep the truncation error satisfactorily small. Moreover, the time step size $\Delta\tau$ is also limited by the fact that a quantity

Table II. Admissable Order for Computing Optimal Expected-Value Functions.

m_1	m_2	n	m_1	m_2	n
0	0	0	3	1	0
1	0	0	3	2	0
0	1	0	1	3	0
0	0	1	2	3	0
1	1	0	3	3	0
0	1	1	0	3	1
1	0	1	0	3	2
1	1	1	0	1	3
2	0	0	0	2	3
0	2	0	0	3	3
0	0	2	3	0	1
2	1	0	3	0	2
1	2	0	1	0	3
2	2	0	2	0	3
0	2	1	3	0	3
0	1	2	3	1	1
0	2	2	3	2	1
2	0	1	3	1	2
1	0	2	3	2	2
2	0	2	1	3	1
2	1	1	2	3	1
1	2	1	1	3	2
1	1	2	2	3	2
2	2	1	1	1	3
1	2	2	2	1	3
2	1	2	1	2	3
2	2	2	2	2	3
3	0	0	3	3	2
0	3	0	2	3	3
0	0	3	3	2	3
			3	3	3
			4	0	0
			0	4	0
				etc.	

Note: Admissible order is top to bottom, starting with column (composed of m_1, m_2, n) on left.

Table III. General Case for Admissable Order of Computing Optimal Expected-Value Functions.

m_1	m_2	n	m_1	m_2	n
i	0	0	1	0	i
0	i	0	2	0	i
0	0	i	⋮	⋮	⋮
i	1	0	(i-1)	0	i
i	2	0	i	0	i
⋮	⋮	⋮	i	1	1
i	(i-1)	0	i	2	1
1	i	0	⋮	⋮	⋮
2	i	0	i	(i-1)	1
⋮	⋮	⋮	i	1	2
(i-1)	i	0	i	2	2
i	i	0	⋮	⋮	⋮
0	i	1	i	(i-1)	2
0	i	2	i	1	3
⋮	⋮	⋮	⋮	⋮	⋮
0	i	(i-1)	i	(i-1)	(i-1)
0	1	i	1	i	1
0	2	i	2	i	1
⋮	⋮	⋮	⋮	⋮	⋮
0	(i-1)	i	(i-1)	i	(i-1)
0	i	i	1	1	i
i	0	1	2	1	i
i	0	2	⋮	⋮	⋮
⋮	⋮	⋮	(i-1)	(i-1)	i
i	0	(i-1)	i	i	(i-1)
			(i-1)	i	i
			i	(i-1)	i
			i	i	i
			(i+1)	0	0
				etc.	

Note: Admissible order is top to bottom, starting with column (composed of m_1, m_2, n) on left.

like $(\Delta\tau)(b_1m_1+b_2m_2)$ or $a_1n\Delta\tau$ or $a_2n\Delta\tau$ in (33) represents a probability and hence must be less than one. Thus, in computational work we must place the following restrictions on the time step size

$$(b_1m_1+b_2m_2)\Delta\tau < 1,$$

$$a_1n\Delta\tau < 1,$$

$$a_2n\Delta\tau < 1. \tag{34}$$

Moreover, in our computational work (performed by M.S. thesis student, R. Powers, Lt. USN) we have used a time step size which yields agreement in the fourth decimal place to the right of the decimal point when $\hat{S}$ is compared to the exact analytic solution S in the special cases (such as (24) through (32) when the latter has been obtained.

7. <u>Solution Behavior for Large Number of Combatants</u>.

Analytically we have developed a solution to equations (15) through (20) up through $S(\tau,1,1,1)$ (see (24) and (29) through (32); see also Table II for admissible order of computation). Computationally, we can generate a numerical solution up through $S(\tau,10,10,10)$ (computed from (33) and requiring computation in the order, for example, shown in Table III). However, it appears infeasible for computational reasons (depending upon both the amount of core storage and also CPU processing time) to generate

a solution much past force levels of the approximate magnitude $m_1 = 10$, $m_2 = 10$, $n = 10$. Thus, it is of import to consider approximate solution behavior of (15) through (20) for large numbers of combatants.

We will first consider limiting behavior of (15) through (20) by (naively) replacing differences in force levels by their approximations as partial derivatives times the length of the interval (unity)--deliberately ignoring the fact that $S(t,x_1,x_2,y)$ had originally been defined only for integer values of x_1, x_2 and y. We seek to approximate our problem with a discrete state space by one in which the state of our system (i.e. numbers of X_1, X_2, and Y combatants) varies continuously. Markov processes in which only continuous changes in state occur are called diffusion processes (see p. 203 of [11]). We will show that the naive passage in the limit leads to an inappropriate equation (one which ignores the probabilistic nature of the process). Later we shall give the parabolic partial differential equation (see pp. 57-77 of [22]) which we feel yields the appropriate approximation to the fundamental equation (15).

Diffusion approximations to Markov processes with discrete states have been extensively studied the past 30 years (see survey article by Iglehart [26], sections in [2], [11], and [18], and the papers [19], [23], and [28]). We will give some of our preliminary results below. Our arguments will be formal, with a

rigorous justification being beyond the scope of our present investigation. (In such matters a rigorous justification requires delicate and subtle arguments and is itself a topic of contemporary mathematical research.) The idea of considering diffusion approximations to Lanchester-type stochastic processes is due to D. Gaver. (Of course, all inadequacies of the present presentation are my own responsibility.) Diffusion approximations have proven to be very useful in genetics [17], [28], queueing [23], and other areas of applications of stochastic processes.

Let us first exhibit the unsatisfactory approximation to (15). The first thing that we will do is to replace τ by $T - t$. Next, we replace differences in force levels, for example, as follows

$$S(t,m_1,m_2,n-1) - S(t,m_1,m_2,n) = \{(y-1)-y\} \frac{\partial S}{\partial y}(t,x_1,x_2,y) + O((\Delta y)^2), \quad (35)$$

where $\Delta y = 1$. Now, for large numbers of combatants we may ignore the integer restrictions on the X_1-, X_2-, and Y- force levels to obtain the following approximation to (15)

$$\frac{\partial S}{\partial t} - (b_1x_1+b_2x_2) \frac{\partial S}{\partial y} + \underset{0 \leq \phi \leq 1}{\text{maximum}} \left\{\phi a_1 y \left(- \frac{\partial S}{\partial x_1}\right) + (1-\phi) a_2 y \left(- \frac{\partial S}{\partial x_2}\right)\right\} = 0, \quad (36)$$

where $S = S(t,x_1,x_2,y)$. The above is precisely the Hamilton-Jacobi-Isaacs equation for the deterministic optimal control problem (see Appendix G). Similar approximations for the Lanchester stochastic process have been given by Willard [45], Koopman [29], and Etter [15].

We will show below, however, that the above approximation is inappropriate, since the corresponding attrition process is deterministic and we have thusly destroyed the probabilistic nature of the attrition process.

We now show the equivalence of (36) with some well-known results from the optimal control of the deterministic Lanchester attrition process. Clearly, (36) yields the maximum principle

$$\underset{0 \leq \phi \leq 1}{\text{maximum}} \; H(t, x_i^*, p_i^*, \phi), \tag{37}$$

where

$$H(t, x_i, p_i, \phi) = -\phi a_1 y p_1 - (1-\phi)\, a_2 y p_2 - (b_1 x_1 + b_2 x_2)\, p_3,$$

and

$$p_i^*(t) = \frac{\partial S}{\partial x_i}(t, x_1, x_2, y) \quad \text{for} \quad i = 1,2,$$

$$p_3^*(t) = \frac{\partial S}{\partial y}(t, x_1, x_2, y).$$

Denoting the value of ϕ which yields the maximum in (37) by ϕ^*, we obtain the following partial differential equation for $S(t, x_1, x_2, y)$

$$\frac{\partial S}{\partial t} - (b_1 x_1^* + b_2 x_2^*) \frac{\partial S}{\partial y} - \phi^* a_1 y^* \frac{\partial S}{\partial x_1} - (1-\phi^*) a_2 y^* \frac{\partial S}{\partial x_2} = 0, \tag{38}$$

which is readily solved by the method based on characteristics (see [22], especially pp. 44-49) to yield

$$\frac{dx_1^*}{dt} = -\phi^* a_1 y^*,$$

$$\frac{dx_2^*}{dt} = -(1-\phi^*)a_2 y^*,$$

$$\frac{dy^*}{dt} = -(b_1 x_1^* + b_2 x_2^*),$$

and $$\frac{dp_i^*}{dt} = -\frac{\partial H}{\partial x_i}(t, x_i^*, p_i^*, \phi^*) \quad \text{for} \quad i = 1,2,3, \tag{39}$$

where, for convenience, we have let $x_3 = y$. By the above, we see that (36) apparently corresponds to a deterministic attrition process.

To more fully justify our claim that (36) is an inappropriate diffusion approximation, it is necessary to review some material from the theory of stochastic processes. We do this in the subsections below. First, we discuss a diffusion approximation to a simple random walk. Next, we apply such considerations to the Lanchester stochastic process. Finally, we present our diffusion approximation to the fundamental functional equation (15) for the optimal expected value function. The diffusion approximation to the Lanchester stochastic process is a new result, previously not appearing in the literature.

7.1. Diffusion Approximation to a Simple Random Walk.

It seems appropriate to begin by giving our own treatment of the diffusion approximation to the simple random walk given by

Feller (see pp. 323-327 in [18]). Our brief treatment is based on the theory of first order partial differential equations (see pp. 18-23 of [22]) (we refer the reader to Feller's lucid treatment [18] for a development based on considerations of random variables) and explicitly shows that a naive passage in the limit can destroy the probabilistic nature of the model.

Let us therefore consider an unrestricted random walk starting at the origin. We let $p_{x,n}$ denote the probability that the particle is at x at time n. During each time period, the particle has a probability p of moving one unit to the left and q of moving one unit to the right (where $p + q = 1$). The following partial difference equation describes this random walk

$$p_{x,n+1} = p\, p_{x-1,n} + q\, p_{x+1,n}, \tag{40}$$

with initial conditions

$$p_{0,0} = 1 \ , \ p_{x,0} = 0 \quad \text{for} \quad x \neq 0.$$

Let us write (40) as

$$p(x,n+1) = p\, p(x-1,n) + q\, p(x+1,n), \tag{41}$$

with initial conditions

$$p(0,0) = 1, \ p(x,0) = 0 \quad \text{for} \quad x \neq 0.$$

We observe that the time parameter is discrete (i.e. $n = 0,1,2,\ldots$) and the state space is discrete (i.e. $x = -\infty,\ldots,-2,-1,0,1,2,\ldots$).

In developing a diffusion (Markov process with continuous state space [11] approximation to (41) we let the time parameter have increments of Δt and the state space have increments of Δx and pass in the limit as $\Delta t \to 0$ and $\Delta x \to 0$. Then, (41) may be re-written as

$$p(x,t+\Delta t) = p\, p(x-\Delta x,t) + q\, p(x+\Delta x,t). \tag{42}$$

If we approximate (42) by retaining only first order terms, we obtain

$$\frac{\partial p}{\partial t}\Delta t = (\Delta x)(q-p)\frac{\partial p}{\partial x}. \tag{43}$$

Letting $\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} (q-p)\left(\frac{\Delta x}{\Delta t}\right) = -2c,$

we may obtain from (43)

$$\frac{\partial p}{\partial t} + 2c\frac{\partial p}{\partial x} = 0, \tag{44}$$

with initial condition

$$p(x,t=0) = \delta(x),$$

or, more generally,

$$p(x,t=0) = \delta(x-x_0),$$

where $\delta(x)$ denotes the delta function of P. Dirac. Equation (44) is our naive approximation to the random walk described by (40). The quantity $p(x,t)$ is the probability density of the particle, i.e.

$$p(x,t)dx = \text{Prob[particle between } x \text{ and } x+dx \text{ at } t].$$

The solution to (44) is readily obtained by characteristics [22] to be

$$\frac{dt}{1} = \frac{dx}{2c} = \frac{dp}{0},$$

or

$$p(x,t) = \text{constant for } \frac{dx}{dt} = 2c,$$

which we may write as

$$p(x,t) = \delta(x-x_0-2ct). \tag{45}$$

Thus, our random walk has become a completely deterministic process. We may think of this as that all probability remains concentrated at a point which moves in a strictly deterministic fashion.

Feller (pp. 323-327 in [18]) shows that an approximation to (41) which preserves the probabilistic nature of the random walk process is given by

$$\frac{\partial p}{\partial t} = -2c\frac{\partial p}{\partial x} + D\frac{\partial^2 p}{\partial x^2}, \tag{46}$$

with initial condition

$$p(x,t=0) = \delta(x-x_0).$$

Equation (46) is a parabolic partial differential equation with solution

$$p(x,t) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left\{-\frac{(x-x_0-2ct)^2}{4Dt}\right\}. \tag{47}$$

In (46), $2c$ is termed the infinitesimal mean (or drift) and $2D$ is the infinitesimal variance. We observe that (44) is a special case of (46) with an infinitesimal variance of zero, i.e. $D = 0$. Hence, our statement that (44) represents a purely deterministic process.

Additionally, it seems appropriate to point out that there is a gap in Feller's argument on p. 325 of [18]. We feel that he should have said that to obtain reasonable results one must let Δx and Δt approach zero in such a way that the limit of the variance of the total displacement in time t is a finite, non-zero quantity. Feller neglects to state the key aspect that the infinitesimal variance be non-zero. Otherwise, as we saw above, one can easily erroneously obtain an approximating equation which may be thought of as representing a purely deterministic process with no random fluctuations in the sample path.

Equation (46) is a special case of the forward Kolmogorov equation (also called the Fokker-Planck equation)

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \{\alpha(x)p\} - \frac{\partial}{\partial x} \{\beta(x)p\}, \tag{47}$$

where

$p = p(t,x;t_0,x_0)$ is the transition probability density,

i.e. $p(t,x;t_0,x_0)\,dx = \text{Prob}\left[\begin{array}{l|l} \text{particle between } x & \text{particle between } x_0 \\ \text{and } x + dx \text{ at } t & \text{and } x_0 + dx_0 \text{ at } t_0 \end{array}\right]$

We recall that the position of the particle is a random variable, denoted by $X(t)$. Also, $\underline{\beta(x)}$ is the <u>infinitesimal</u> <u>mean</u> and is given by

$$\beta(x) = \lim_{\Delta t \to 0} \frac{E[\{x(t+\Delta t)-x(t)\} \mid x(t)=x]}{\Delta t}, \tag{48}$$

where $E[X]$ denotes the expectation of the random variable X. Furthermore, $\underline{\alpha(x)}$ is the <u>infinitesimal</u> <u>variance</u> and is given by

$$\alpha(x) = \lim_{\Delta t \to 0} \frac{V[\{x(t+\Delta t)-x(t)\} \mid x(t)=x]}{\Delta t}, \tag{49}$$

where $V[X]$ denotes the variance of the random variable X. (The definitions (48) and (49) involve conditional mean and conditional variance.) The process corresponding to (47) is stationary, since α and β do not depend upond time. The specification of the infinitesimal mean and variance determines the Markov process (uniquely).

7.2. Diffusion Approximation to the Lanchester Stochastic Process.

A number of researchers [45], [29], [15] have considered (in a heuristic fashion as we do here) the limiting behavior of the Kolmogorov forward equations for the Lanchester stochastic process for large numbers of combatants. However, we feel that the approximation to the combat attrition random process that each developed was inappropriate, since the random nature of the attrition process had been destroyed in the passage to the limit. (There apparently are several unanswered theoretical questions in this area, especially since Etter's development [15] is very precise. It should be noted that Etter's results are obtained by using Lax and Richtmyer's well-known theory (a key aspect being Lax's equivalence theorem (see p. 44 of [36])) on the convergence of finite difference operators to differential operators. This latter theory was developed for deterministic problems in the physical sciences (hydrodynamics, transport theory, etc.) and may require modification for probabilistic systems.)

Before presenting our new results, let us show the development of the (inappropriate) approximation given in [45], [29], [15]. For simplicity, let us consider the Lanchester square-law attrition process for combat between two homogeneous forces. The Kolmogorov forward equations are

for $0 < m < M$, $0 < n < N$

$$\frac{dP}{dt}(m,n,t) = a\ n\ P(m+1,n,t) + b\ m\ P(m,n+1,t) - (an+bm)\ P(m,n,t), \quad (50)$$

where m and n are non-negative integers. Considering large numbers of combatants, it does not seem inappropriate to replace m and n by x and y which no longer have to be integers. We note that when m was restricted to integer values, $m + 1 = m + \Delta m$ and similarly for n. Observe that here we have $\Delta m = \Delta n = 1$. It is convenient to write (50) as

$$\frac{dP}{dt}(m,n,t) = an[P(m+1,n,t) - P(m,n,t)] + bm[P(m,n+1,t) - P(m,n,t)]. \quad (51)$$

Now when we replace m and n by x and y, we may think of (51) as

$$\frac{\partial P}{\partial t} = ay\left[\frac{P(x+\Delta x,y,t)-P(x,y,t)}{\Delta x}\right] + bx\left[\frac{P(x,y+\Delta y,t)-P(x,y,t)}{\Delta y}\right],$$

where we allow fractional casualties in our formulation. Letting $\Delta x,\ \Delta y \to 0$, we (formally) obtain

$$ay\ \frac{\partial p}{\partial x} + bx\ \frac{\partial p}{\partial y} - \frac{\partial p}{\partial t} = 0, \quad (52)$$

with initial condition

$$p(x,y,t=0) = \delta(x-x_0)\ \delta(y-y_0),$$

where

$$p(x,y,t)\ dx\ dy = \text{Prob}\left[\begin{array}{l}\text{at } t \text{ the } X \text{ force level is between } x \text{ and} \\ x+dx \text{ and the } Y \text{ force level is between } y \text{ and } y+dy\end{array}\right],$$

i.e. $p(x,y,t)$ is a probability density function.

The solution to (52) is readily obtained by characteristics to be

$$\frac{dx}{ay} = \frac{dy}{bx} = \frac{dt}{-1} = \frac{dp}{0},$$

or

$$p(x,y,t) = \text{ constant, for}$$

$$\frac{dx}{dt} = -\,ay,$$

$$\frac{dy}{dt} = -\,bx. \tag{53}$$

We observe that (53) are just the deterministic Lanchester equations corresponding to (50). Using the initial condition $p(x,y,t=0) = \delta(x-x_0)\,\delta(y-y_0)$, it is easy to show that (53) holds for the average force levels, i.e.

$$\frac{d\overline{x}}{dt} = -\,a\overline{y},$$

$$\frac{d\overline{y}}{dt} = -\,b\overline{x}, \tag{54}$$

where $\overline{x}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, p(x,y,t)\, dx\, dy$, etc.. We observe that the system average is identical with the sample path. Thus, the limit process (52) is inappropriate for approximating the Lanchester stochastic process (50), since (52) is the equation for a deterministic process.

Thus, we seek an approximation which preserves the random nature of the attrition process. Following W. Feller (see p. 235 of [17]; see also p. 235 of [11] and [23]), we will construct an approximation to a stochastic process with discrete states by considering a corresponding Markov process with continuous state space, which has an equivalent infinitesimal mean and variance.

We begin by considering a battle with discrete time steps. Over each time step, the number of casualties that a Y soldier produces is a random variable with mean μ_a and variance σ_a^2 and similarly for X. We let

$$X_n = X\text{-force level at time step } n \text{ (a random variable)},$$

$$Y_n = Y\text{-force level at time step } n \text{ (a random variable)}.$$

Then

$$E[X_{n+1}-X_n | X_n=x, Y_n=y] = - y\mu_a,$$

and

$$V[X_{n+1}-X_n | X_n=x, Y_n=y] = y\sigma_a^2.$$

This suggests approximating the infinitesimal means and variance as follows:

$$\beta_X = - a_1 y \quad \text{and} \quad \alpha_X = Ay,$$

$$\beta_Y = -\,b_1 x \quad \text{and} \quad \alpha_Y = Bx, \tag{55}$$

where a_1, b_1, A, $B > 0$. Then the equation for $p(x,y,t)$ would be

$$\frac{1}{2}\left\{Ay\,\frac{\partial^2 p}{\partial x^2} + Bx\,\frac{\partial^2 p}{\partial y^2}\right\} + a_1 y \frac{\partial p}{\partial x} + b_1 x\,\frac{\partial p}{\partial y} = \frac{\partial p}{\partial t}, \tag{56}$$

with initial condition $p(x,y,t=0) = S(x-x_0)\ S(y-y_0)$.

We observe that as long as $A > 0$ and $B > 0$ the probability density diffuses over time. For $A = 0$ and $B = 0$, we have a deterministic process. Equation (56) is the diffusion approximation to the Lanchester (square-law) stochastic process and is the major result of this section. Unfortunately, (56) is not readily solved by the method of characteristics as was the first order partial differential equation (52).

7.3. Diffusion Approximation to Fundamental Functional Equation.

Based on our consideration of diffusion approximations in Sections 7.1 and 7.2 above, a diffusion approximation to the fundamental functional equation (15) is easily written down. Recalling that $\tau = T - t$ is "backwards time" and considering the retrospective (backward) evolution of the system from end of battle $t = T$ (or equivalently $\tau = 0$), the appropriate approximation is given by

$$\frac{\partial S}{\partial \tau}(\tau,x_1,x_2,y) = -(b_1x_1+b_2x_2)\frac{\partial S}{\partial y} + \frac{1}{2}(B_1x_1+B_2x_2)\frac{\partial^2 S}{\partial y^2}$$

$$+ \underset{0 \le \phi \le 1}{\text{maximum}} \left\{ \phi a_1 y\left(-\frac{\partial S}{\partial x_1}\right) + (1-\phi)a_2 y\left(-\frac{\partial S}{\partial x_2}\right)\right\}$$

$$+ \frac{1}{2}A_1 y\frac{\partial^2 S}{\partial x_1^2} + \frac{1}{2}A_2 y\frac{\partial^2 S}{\partial x_2^2}, \qquad (57)$$

with terminal condition

$$S(\tau=0,x_1,x_2,y) = ry(T) - px_1(T) - qx_2(T).$$

In (57) we have the backward operator (using "backward time") for the continuous Markov process (see [20]).

8. Proposed Future Research.

The results presented in this appendix are preliminary and should be expanded in two directions: (1) computational and (2) theoretical. We propose both to ONR as future research tasks, with the computational investigations deserving the higher priority.

The most important future task is to generate some numerical results via the finite difference approximation (33). As noted above, the adequacy of the finite difference approximation (in particular, the time step size) can be checked by considering the (partial) exact analytic solution that has been developed. A M. S. thesis student (R. Powers, Lt., USN) is currently performing this computational investigation. (A previous investigation by

S. Silvasy, Maj., USA, proved to be inconclusive.) Results of computations for the optimal control of the Lanchester stochastic process can then be compared with results on the optimal control of the deterministic Lanchester process. The goal of this proposed investigation is to determine the effect of the nature of the attrition model (deterministic or stochastic) on the structure of the optimal target selection policy. Additionally, it is of interest to know if the optimal policy for the deterministic process is valid for small numbers of combatants and a stochastic attrition mechanism.

Theoretical investiations should center about gaining more insight into the structure of the optimal target selection policy when the attrition mechanism is stochastic. Topics such as steady-state behavior of equation (15), further development of the analytic solution, development of an analytic solution to the diffusion approximation (57), etc. merit further investigation. Additionally, the theoretical justification of passage to the limit for the birth and death equations should be investiated further.

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Appendix J. Inertial Combat.

1. Introduction.

In all the target selection problems that we have considered previously in this report an implicit assumption has always been that fire could be instantaneously shifted from one target type to another. To illustrate, let us recall a typical problem:

$$\max_{\phi(t)} \{ry(T) - px_1(T) - qx_2(T)\},$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2,$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1. \tag{1}$$

In this problem ϕ is the control (decision or policy) variable. The reader should note that although the control must satisfy the condition $0 \leq \phi \leq 1$, the rate of change of ϕ is unrestricted so that ϕ can instantaneously change, for example, from 0 to 1. Physically, this means that we are assuming that the Y-forces can instantaneously shift fires as desired.

When one considers command and control problems in combat, the above implicit assumption on ϕ (instantaneous jumps permitted) does not seem to be a realistic assumption. A better assumption appears to be that there is a limit to how fast ϕ can be changed. We

have thus been led to consider target selection problems in which the rate of change of the allocation variable is bounded (i.e. instantaneous shifts in fire are not allowed). For reasons discussed below, we have chosen to call such a situation "inertial combat."

Although problems in which curves are restricted to lie in a given domain were considered in the classical calculus of variations as long ago as 1831 [3] (see also [1]) and discussed by Weierstrass in his lectures of 1879 (see p. 395 of [1]), development of optimality conditions for optimal control problems with state variable inequality constraints has been accomplished only comparatively recently. We have discussed such problems with a state variable inequality constraint (SVIC) in Appendix E. Recent activity apparently owes its origin to the work of Gamkrelidze (for an English translation of his original work see Chapter VI of [8]). Gamkrelidze points out that in many physical problems there are restrictions not only on the control parameters but also on the state (phase) space. He (see p. 263 of [8]) refers to piecewise continuous controls as "inertialess controls," since such controls can, if need be, instantaneously jump from one value to another. Following Gamkrelidze then, we use the term inertial combat to refer to a target selection problem in which the rate of change of the distribution of fire is bounded.

The problem that we shall study requires more than the theory of Gamkrelidze (see Appendix E for further discussion and references),

which, of course, is for optimal control problems involving inequality constraints on a function of the state variables with no explicit dependence on the control variables. We recall that we say that the problem has a $k^{\underline{th}}$ order SVIC when the first time (total) derivative of the state-variable constraint which explicitly contains the control variables is the $k^{\underline{th}}$. The problem that we shall study has a second order SVIC. Unfortunately, there apparently has not been developed any condition analogous to, using the notation of Appendix E, $\dot{\mu} \leq 0$ for a $k^{\underline{th}}$ order SVIC $(k > 1)$. Using the method of adjoining the state-variable constraint directly to the criterion functional with an additional Lagrange multiplier, we will develop a necessary condition of optimality on a constrained subarc that corresponds to, again using the notation of Appendix E, having a restriction on $\ddot{\mu}$. This latter condition apparently arises because of the problem's special structure and does not appear to be a general condition.

The method of directly adjoining the state-variable constraint was apparently first considered by Chang [2]. However, there are some gaps in his development. A more careful treatment is by Speyer and Bryson [10]. McIntyre and Paiewonsky [7] also discuss the method of direct adjoining of the SVIC (and its relationship to the method of adjoining the appropriate time derivative of the SVIC). It may be shown that results obtained by the method of direct adjoining of the state-variable constraint correspond to

those obtained by direct application of the Kuhn-Tucker conditions in a Banach space (i.e. consider a finite difference approximation to a variational problem (a finite-dimensional optimization problem) and then pass to the limit).

The results presented in this appendix are preliminary in nature and tentative, obtained from our first cursory examination of such a problem. However, we feel that this is a promising area for obtaining results and insights into principles of target selection. We propose to ONR further examination of such problems as a future research task.

2. The Optimal Control Problem.

Accordingly, we consider the following problem:

$$\max_{u(t)} \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with } T \text{ specified,}$$

$$\text{subject to: } \frac{dx_1}{dt} = -\phi a_1 y,$$

$$\frac{dx_2}{dt} = -(1-\phi)a_2 y,$$

$$\frac{dy}{dt} = -b_1x_1 - b_2x_2,$$

$$\frac{d\phi}{dt} = u, \tag{2}$$

$$x_1,\ x_2,\ y \geq 0, \quad 0 \leq \phi \leq 1, \quad \text{and} \quad -A \leq u \leq B,$$

where

$x_1(t)$, $x_2(t)$, $y(t)$ are force levels,

p, q, r are utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

ϕ is the fraction of Y fire directed at X_1,

and u is the rate of change of ϕ.

We will focus primarily on the development of the basic necessary conditions of optimality for (2). For this goal, of course, the nature of the planning horizon (terminal target set or prescribed duration) doesn't make any difference.

The reader should note that the control variable in problem (2) is u, while ϕ (the control variable in problem (1)) is now a <u>state variable</u>. Hence, the restriction $0 \leq \phi \leq 1$ is now a (first order) SVIC. When we use the approach of Gamkrelidze, we handle it as follows:

for $\phi - 1 \leq 1$, we must have

$$\frac{d\phi}{dt} = u \leq 0 \quad \text{when} \quad \phi - 1 = 0, \tag{3}$$

and

for $-\phi \leq 0$, we must have

$$-\frac{d\phi}{dt} = -u \leq 0 \quad \text{when} \quad -\phi = 0. \tag{4}$$

To avoid being encumbered by too many symbols, we will consider only the SVIC $x_1 \geq 0$. Clearly, we lose no generality in doing so. In the notation of Appendix E, we have then

$$C\ (t,x) = -\ x_1 \leq 0, \tag{5}$$

$$\frac{dC}{dt} = -\frac{dx_1}{dt} = \phi a_1 y, \tag{6}$$

$$\frac{d^2C}{dt^2} = ua_1 y - \phi a_1(b_1x_1+b_2x_2). \tag{7}$$

Now, when $x_1(t) = 0$ for $t_1 \leq t \leq t_2 = T$, we control the system by requiring

$$C\ (t,x) = -\ x_1 = 0, \tag{8}$$

$$\frac{dC}{dt}(t,x) = -\frac{dx_1}{dt} = \phi a_1 y = 0, \tag{9}$$

and

$$\frac{d^2C}{dt^2} = ua_1 y - \phi a_1(b_1x_1+b_2x_2) \leq 0. \tag{10}$$

Hence, the origin of the term second order SVIC. Clearly, when $x_1(t) = 0$ for a finite interval of time, by (9) we must have $\phi^*(t) = 0$ (since $y > 0$) and then (10) yields $u^*(t) = 0$, where we have considered the state equations (2).

3. Necessary Conditions of Optimality on Constrained Subarc for ϕ.

When Gamkrelidze's approach (see Chapter VI of [8]) and considers (3), (4), (8), (9), and (10), then the Hamiltonian is given by

$$H(t,\ x_i, p_i, u) = -\ p_1\phi a_1 y - p_2(1-\phi)a_2 y - p_3(b_1x_1+b_2x_2)$$

$$+\ p_4u - \eta_1(t)u + \eta_2(t)u - \mu(t)\{ua_1y - \phi a_1(b_1x_1+b_2x_2)\}, \qquad (11)$$

where

$$\eta_1(t) \begin{cases} = 0 & \text{for} \quad \phi < 1, \\ \geq 0 & \text{for} \quad \phi = 1, \end{cases}$$

$$\eta_2(t) \begin{cases} = 0 & \text{for} \quad \phi > 0, \\ \geq 0 & \text{for} \quad \phi = 0, \end{cases}$$

and

$$\mu(t) \begin{cases} = 0 & \text{for} \quad x_1 > 0, \\ \geq 0 & \text{for} \quad x_1 = 0. \end{cases}$$

We have adopted above the following correspondence between state and dual variables:

state variable	dual variable
x_1	p_1
x_2	p_2
y	p_3
ϕ	p_4.

The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = b_1p_3 - \mu(t)\phi a_1 b_1, \qquad (12)$$

$$\frac{dp_2}{dt} = b_2p_3 - \mu(t)\phi a_1b_1, \tag{13}$$

$$\frac{dp_3}{dt} = \phi a_1p_1 + (1-\phi)a_2p_2 + \mu(t)a_1u, \tag{14}$$

$$\frac{dp_4}{dt} = (a_1p_1 - a_2p_2)y - \mu(t)a_1(b_1x_2+b_2x_2). \tag{15}$$

Assuming that time determines the termination of the battle (planning horizon), i.e. we don't consider termination due to annihilation of one side or the other before $t = T$, the boundary conditions for the dual variable at $t = T$ are given by

$$p_1(t=T) = -p + \nu_1,$$

where

$$\nu_1 \begin{cases} = 0 & \text{for } x_1(T) > 0, \\ \geq 0 & \text{for } x_2(T) = 0, \end{cases} \tag{16}$$

$$p_2(t=T) = -q + \nu_2,$$

where

$$\nu_2 \begin{cases} = 0 & \text{for } x_2(T) > 0, \\ \geq 0 & \text{for } x_2(T) = 0, \end{cases} \tag{17}$$

$$p_3(t=T) = r + \nu_3,$$

where

$$\nu_3 \begin{cases} = 0 & \text{for } y(T) > 0, \\ \geq 0 & \text{for } y(T) = 0, \end{cases} \tag{18}$$

$$p_4(t=T) = \nu_4 - \nu_5$$

where

$$\nu_4 \begin{cases} = 0 & \text{for} \quad \phi(T) > 0, \\ \geq 0 & \text{for} \quad \phi(T) = 0, \end{cases}$$

and

$$\nu_5 \begin{cases} = 0 & \text{for} \quad \phi(T) < 1, \\ \geq 0 & \text{for} \quad \phi(T) = 1. \end{cases} \tag{19}$$

When $x_1, x_2, y > 0$ and $0 < \phi < 1$, the control law is determined by Pontryagin's maximum principle. Hence, we consider

$$\underset{-A \leq u \leq B}{\text{maximize}} \quad H(t, x_i, p_i, u),$$

and this yields

$$u^*(t) = \begin{cases} B & \text{for} \quad p_4(t) > 0, \\ -A & \text{for} \quad p_4(t) < 0. \end{cases} \tag{20}$$

We must further investigate the possibility of a singular subarc [5], [6] on which $\frac{\partial H}{\partial u} = 0$ for a finite interval of time (so that all time derivatives vanish). The condition that $\frac{\partial H}{\partial u} = 0$ yields that on a singular subarc we must have

$$p_4(t) = 0. \tag{21}$$

The condition $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ then yields

$$a_1 p_1(t) = a_2 p_2(t). \tag{22}$$

Proceeding to the next time derivative, we would have on the singular subarc when (22) holds that

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = yp_3(a_1b_1 - a_2b_2). \tag{23}$$

Considering (23), we see that a singular solution is impossible.

On a constrained subarc on which $\phi(t) = 1$ for $t_1 \leq t \leq t_2$ the control is determined by $\frac{d\phi}{dt} = 0$ and hence

$$u^*(t) = 0 \quad \text{for} \quad t_1 < t < t_2. \tag{24}$$

The multiplier $\eta_1(t)$ is determined by the condition $\frac{\partial H}{\partial u} = 0$ and hence

$$\eta_1(t) = p_4(t). \tag{25}$$

The condition that $\eta_1(t) \geq 0$ yields that on the constrained subarc we must have

$$p_4(t) \geq 0. \tag{26}$$

Differentiating (25) and combining with (15), we find that

$$\dot{\eta}_1(t) = y(a_1p_1 - a_2p_2), \tag{27}$$

so that Gamkrelidze's condition $\dot{\eta}_1(t) \leq 0$ [8] is only satisfied on the constrained subarc with $\phi = 1$ when

$$a_1(-p_1(t)) \geq a_2(-p_2(t)), \tag{28}$$

which the reader will, of course, recognize as a result for the corresponding "inertialess" combat problem. Denoting the time of an entrance corner by t_1 and that of an exit corner by t_2, the corner conditions (see Appendix E for discussion) yield that at an entrance corner

$$p_i(t_1-) = p_i(t_1+) \quad \text{for} \quad i = 1,2,3, \tag{29}$$

and

$$p_4(t_1-) = 0 = p_4(t_1+) - \eta_1(t_1+), \tag{30}$$

or

$$p_4(t_1+) = \eta_1(t_1+), \tag{31}$$

where t_1- denotes a left-hand limit. The reader should note that (31) is in consonance with (25). Futhermore, at an exit corner we have

$$p_i(t_2-) = p_i(t_2+) \quad \text{for} \quad i = 1,2,3,4. \tag{32}$$

Considering either $\eta_1(t_2-) = 0$ or $H(t_2-) = H(t_2+)$, we readily find that

$$p_4(t_2-) = 0 = p_4(t_2+). \tag{33}$$

Considering (27), (31), and again $\eta_1(t_2-) = 0$, we see that when there is an exit from the constrained subarc at $t = t_2$, then (31) becomes

$$p_4(t_1+) = \eta_1(t_1+) = \int_{t_1}^{t_2} y(a_2p_2 - a_1p_1)\, dt \geq 0. \tag{34}$$

On a constrained subarc on which $\phi(t) = 0$ for $t_1 \leq t \leq t_2$ the control is determined by $\frac{d\phi}{dt} = 0$ and hence

$$u^*(t) = 0 \quad \text{for} \quad t_1 < t < t_2. \tag{35}$$

The multiplier $\eta_2(t)$ is determined by the condition $\frac{\partial H}{\partial u} = 0$ and hence

$$\eta_2(t) = - p_4(t). \tag{36}$$

The condition that $\eta_2(t) \geq 0$ yields that on the constrained subarc we must have

$$p_4(t) \leq 0. \tag{37}$$

Differentiating (36) and using (15), we find that

$$\dot{\eta}_2(t) = y(a_2p_2 - a_1p_1), \tag{38}$$

so that Gamkrelidze's condition $\dot{\eta}_2(t) \leq 0$ is only satisfied on the constrained subarc with $\phi = 0$ when

$$a_1(-p_1(t)) \leq a_2(-p_2(t)), \tag{39}$$

which again the reader will recognize as a result which also arose in the corresponding "inertialess" combat problem. At an entrance corner, the corner conditions yield

$$p_i(t_1-) = p_i(t_1+) \quad \text{for} \quad i = 1,2,3, \tag{40}$$

and

$$p_4(t_1-) = 0 = p_4(t_1+) + \eta_2(t_1+), \tag{41}$$

or

$$p_4(t_1+) = -\eta_2(t_1+). \tag{42}$$

Again, we point out to the reader that (42) is in consonance with (36). Furthermore, at an exit corner we have

$$p_i(t_2-) = p_i(t_2+) \quad \text{for} \quad i = 1,2,3,4. \tag{43}$$

Considering either $\eta_2(t_2-) = 0$ or $H(t_2-) = H(t_2+)$, we readily find that

$$p_4(t_2-) = 0 = p_4(t_2+). \tag{44}$$

Considering (38), (42), and again $\eta_2(t_2-) = 0$, we see that when there is an exit from the constrained subarc at $t = t_2$, then (42) becomes

$$p_4(t_1-) = -\eta_2(t_1+) = \int_{t_1}^{t_2} y(a_2p_2 - a_1p_1)dt \leq 0. \tag{45}$$

4. <u>Partial Synthesis of Optimal Policy When $x_1, x_2 \geq 0$.</u>

In this section, we will give some preliminary results on how the necessary conditions of the last section may be used to determine the optimal policy. We recall that we developed those results for $x_1, x_2 > 0$.

Since we develop the solution to this problem by working backwards from the end $t = T$, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T - t$. We observe that $\frac{d}{dt} = -\frac{d}{d\tau}$ but $\frac{d^2}{dt^2} = \frac{d^2}{d\tau^2}$. It is also convenient to define

$$v(t) = a_1p_1(t) - a_2p_2(t), \tag{46}$$

so that differentiating (46) and combining with (12) and (13), we find that

$$\frac{dv}{dt} = p_3(t)(a_1b_1 - a_2b_2). \tag{47}$$

<u>Let us now assume</u> (without any loss of generality) <u>that</u> $\underline{a_1b_1 \geq a_2b_2}$. It is easy to show that $p_3(t) > 0$ for all t and hence

$$\frac{dv}{dt}(t) > 0 \quad \text{for all} \quad t. \tag{48}$$

Using (46), it is convenient to write (15) (for $x_1 > 0$) as

$$\frac{dp_4}{dt} = yv, \tag{49}$$

and hence

$$\frac{d^2p_4}{dt^2} = -(b_1x_1+b_2x_2)\, v + y\,\frac{dv}{dt}. \tag{50}$$

In synthesizing an optimal trajectory there are two cases to be considered:

Case (a) $\quad a_1p \geq a_2q,$

Case (b) $a_1 p < a_2 q$.

For Case (a): $a_1 p \geq a_2 q$, it is convenient to first observe that a Taylor series expansion of $p_4(\tau)$ yields

$$p_4(\tau) = p_4(\tau=0) + \tau \frac{dp_4}{d\tau}(\tau=0) + \frac{\tau^2}{2} \frac{d^2 p_4}{d\tau^2}(\tau=\hat{\tau}), \tag{51}$$

where $\hat{\tau} \in (0,\tau)$.

In this case we have

$$v(\tau=0) = -a_1 p + a_2 q \leq 0, \tag{52}$$

so that considering (48) it is readily seen that

$$v(\tau) < 0 \quad \text{for} \quad \tau > 0, \tag{53}$$

and hence (49) and (50) readily yield

$$\frac{dp_4}{d\tau}(\tau=0) = y\,(a_1 p - a_2 q) \geq 0, \tag{54}$$

$$\frac{d^2 p_4}{d\tau^2}(\tau) > 0. \tag{55}$$

Now, there are three subcases to be considered when $a_1 p \geq a_2 q$:

Subcase (a1) $\phi(t=T) = 0$,

Subcase (a2) $0 < \phi(t=T) > 1$,

Subcase (a3) $\phi(t=T) = 1$.

We shall now show that subcase (a1) is inconsistent with an optimal policy and work out the solution for the other two cases.

<u>Subcase (a1)</u> <u>$\phi(t=T) = 0$</u> <u>when</u> <u>$a_1 p \geq a_2 q$</u>.

Since $\phi(t=T) = 0$, (19) yields that

$$p_4(\tau=0) = \nu_4 \geq 0. \tag{56}$$

Then (51), (54), (55), and (56) yield that

$$p_4(\tau) \geq 0 \quad \text{for} \quad \tau \geq 0, \tag{57}$$

with strict inequality holding, i.e. $p_4(\tau) > 0$, for $\tau > 0$. If we were on a constrained subarc for a finite interval of time, i.e. $\phi(t) = 0$ for $t_1 \geq t \geq T$, then (57) is inconsistent with (37) (also note that $\dot{\eta}_2(t=T) = y(a_1 p - a_2 q) \geq 0$, a violation of Gamkrelidze's condition $\dot{\eta}_2(t) \leq 0$ (when $a_1 p > a_2 q$)). If we were not on a constrained subarc for a finite interval of time, we again reach a contradiction. To see this, let us observe that (20) and (57) yield that

$$u^*(\tau) = B \quad \text{for} \quad \tau \geq 0, \tag{58}$$

so that a backward integration of the state equation (2) yields that

$$\phi(\tau) = - B\tau,$$

which is impossible. Hence this case is inconsistent with an optimal policy.

<u>Subcase (a2)</u> <u>$0 < \phi(t=T) < 1$</u> <u>when</u> <u>$a_1p \geq a_2q$</u>.

Since $0 < \phi(t=T) < 1$, (19) yields $p_4(\tau=0) = 0$, and (51), (54), and (55) again yield (57). Hence, (58) again holds. Denoting $\phi(t=0)$ by ϕ_0, it is easily seen that

$$\phi_0 + BT = \phi(t=T) < 1, \tag{59}$$

so that this case happens when

$$T < \left(\frac{1-\phi_0}{B}\right). \tag{60}$$

The optimal policy is then given by

$$u^*(t) = B \quad \text{for} \quad 0 \leq t \leq T < \left(\frac{1-\phi_0}{B}\right). \tag{61}$$

For larger time T, we must go to the next subcase.

<u>Subcase (a3)</u> <u>$\phi(t=T) = 1$</u> <u>when</u> <u>$a_1p \geq a_2q$</u>.

Since $\phi(t=T) = 1$, (19) now yields

$$p_4(\tau=0) = -\nu_5 \leq 0. \tag{62}$$

It may be shown that we get a contradiction unless $\nu_5 = 0$ so that we must have

$$p_4(\tau=0) = 0. \tag{63}$$

If $\phi(t) < 1$ for $T - \delta \le t < T$ where $\delta > 0$, then the development of the previous subcase holds. If we are on a constrained subarc for $t_1 \le t \le T$, then (recalling (46)) (53) is equivalent to satisfying (28) (Gamkrelidze's condition $\dot{\eta}_1 \le 0$). Hence, we can remain (in our backwards progression from the end $t = T$) on the constrained subarc until we have to get off to meet the initial condition $\phi(t=0) = \phi_0$. As we work backwards and leave the constrained subarc (i.e. in forwards time, enter at t_1), the corner condition (30) yields that for $0 \le t \le t_1$

$$p_4(t) = -(t_1-t)\frac{dp_4}{dt}(t=t_1) + \frac{(t_1-t)^2}{2}\frac{d^2p_4}{dt^2}(t=\hat{t}), \tag{64}$$

where $\hat{t} \in (t,t_1)$.

Recalling (54) and (55), we see that $p_4(t) \ge 0$ for $0 \le t \le t_1$ so that the rest of the analysis is similar to the preceeding subcase. Hence,

for $T \ge \left(\frac{1-\phi_0}{B}\right)$, we have

$$u^*(t) = \begin{cases} B & \text{for } 0 \le t \le \left(\frac{1-\phi_0}{B}\right), \\ 0 & \text{for } \left(\frac{1-\phi_0}{B}\right) < t \le T, \end{cases} \tag{65}$$

$$\phi^*(t) = \begin{cases} \phi_0 + Bt & \text{for} \quad 0 \le t \le \left(\dfrac{1-\phi_0}{B}\right), \\ 1 & \text{for} \quad \left(\dfrac{1-\phi_0}{B}\right) < t \le T. \end{cases} \tag{66}$$

For Case (b): $a_1p < a_2q$, we now have that (recalling (46))

$$v(\tau=0) = -a_1p + a_2q \ge 0. \tag{67}$$

Recalling (48), we see that at some (backwards) time $v(\tau)$ must become zero. Let us denote this "backwards time" as τ_1. Thus,

$$v(\tau=\tau_1) = 0. \tag{68}$$

Again, there are three subcases to be considered when $a_1p < a_2q$:

Subcase (b1) $\phi(t=T) = 0$,

Subcase (b2) $0 < \phi(t=T) < 1$,

Subcase (b3) $\phi(t=T) = 1$.

Analysis of these subcases is similar to that given for Case (a), with Subcase (b3) being impossible.

Let us now observe (recalling (28) and (39)) that in order to satisfy Gamkrelidze's condition on the rate of change on a Lagrange multiplier, we must have

$$v(t) \le 0 \quad \text{when} \quad \phi(t) = 1 \quad \text{for a finite interval of time,} \tag{69}$$

and

$$v(t) \ge 0 \quad \text{when} \quad \phi(t) = 0 \quad \text{for a finite interval of time.} \tag{70}$$

We have previously noted the correspondence of these results to those for "inertialess" combat. We now consider the case when $\phi(t=T) = 0$. Let t_1^o denote the (forward) time when we enter a constrained subarc with $\phi = 0$; similarly t_2^1 the time of leaving one with $\phi = 1$. We further assume that

$$\phi(t) = 0 \quad \text{for} \quad t_1^o \le t \le T. \tag{71}$$

We now prove that it is impossible to have $v(t=t_1^o) = 0$, i.e. $v(t)$ must be > 0 before $\phi = 0$. The corner condition (41) and (49) yield that for $t \le t_1^o$

$$p_4(t) = \frac{(t_1-t)^2}{2} \frac{d^2 p_4}{dt^2}(t=\hat{t}), \tag{72}$$

$$\text{where} \quad \hat{t} \in (t, t_1),$$

so that (recalling that $v(t=t_1^o) = 0$, (48), and (50)) it is readily seen that

$$p_4(t) > 0 \quad \text{for} \quad t < t_1^o. \tag{73}$$

By (20) this implies $u^*(t) = B$ for $t < t_1^{o}$, which clearly is impossible.

Now, it is readily shown that for $t_2^{1} \le t \le t_1^{o}$

$$p_4(t) = -\int_t^{t_1^{o}} y(t)v(t)dt. \tag{74}$$

In particular

$$p_4(t_2^{1+}) = -\int_{t_2^{1}}^{t_1^{o}} y(t)v(t)dt. \tag{75}$$

Considering the corner condition (33) and (48), it is clear that

$$v(t=t_2^{1}) < 0, \tag{76}$$

while

$$v(t=t_1^{o}) > 0. \tag{77}$$

We may also write

$$p_4(t) = \int_{t_2^{1}}^{t} y(t)v(t)dt. \tag{78}$$

Considering (76) through (78), it should be clear that $p_4(t) < 0$ for $t_2^{1} < t < t_1^{o}$ and hence by (20)

$$u^*(t) = -A \text{ for } t_2^{1} < t < t_1^{o}. \tag{79}$$

Integration of the state equation (2) then yields

$$\phi(t) = 1 - A(t-t_2^{\,1}) \quad \text{for} \quad t_2^{\,1} \le t \le t_1^{\,o}, \tag{80}$$

and hence

$$t_1^{\,o} - t_2^{\,1} = \frac{1}{A}. \tag{81}$$

The times $t_2^{\,1}$ and $t_1^{\,o}$ are determined by the conditions

$$T - \tau_1 \in (t_2^{\,1}, t_1^{\,o}), \tag{82}$$

and

$$\int_{t_2^{\,1}}^{t_1^{\,o}} y(t)v(t)dt = 0, \tag{83}$$

or

$$\int_{t_2^{\,1}}^{T-\tau_1} y(t)v(t)dt = -\int_{T-\tau_1}^{t_1^{\,o}} y(t)v(t)dt. \tag{84}$$

The relationship of the times $t_2^{\,1}$, $T - \tau_1$, and $t_1^{\,o}$ to the time history of $\phi(t)$ is shown in Figure 1.

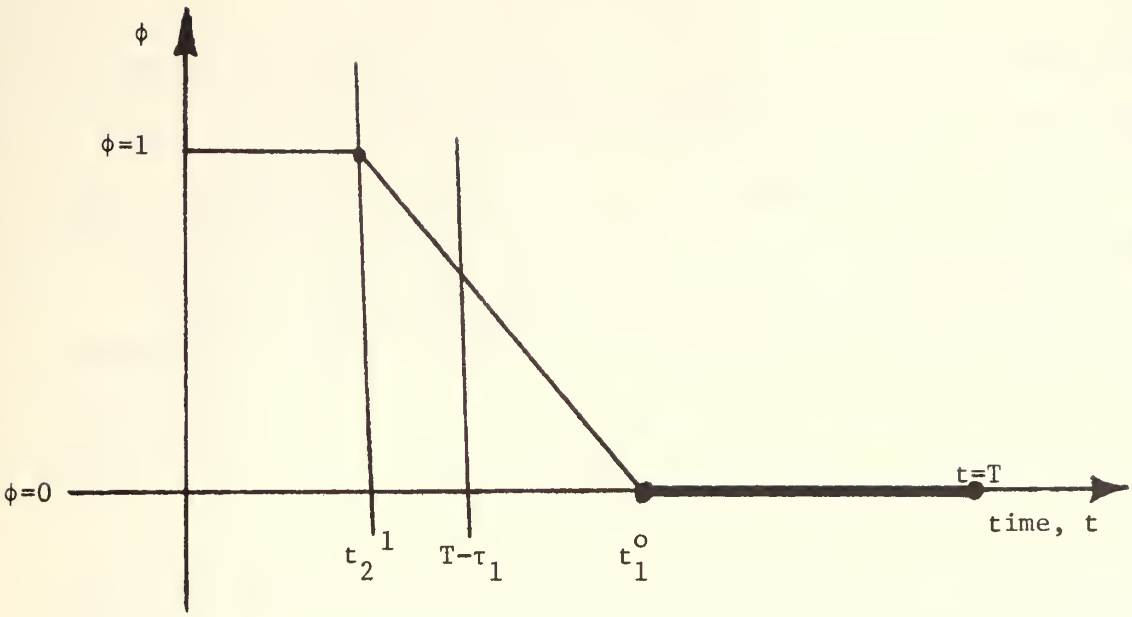

Figure 1. Relationship of $t_2^{\ 1}$, $T - \tau_1$, and $t_1^{\ o}$ to Values of ϕ.

Omitting further details for now, we do reach an important conclusion: for "inertial" combat (see (2)) one begins to shift fire earlier in forward time (anticipating changes in target priority) than in the corresponding "inertialess" case. Again, the reader is referred to Figure 1 for motivation of this statement.

5. Necessary Conditions of Optimality on Constrained Subarc for x_1.

Let us first consider Gamkrelidze's approach of considering the time derivative of the state-variable constraint on a constrained subarc on which $x_1(t) = 0$ (and $x_2 > 0$) for $t_1 \le t \le t_2$. We limit our discussion here to the necessary conditions for it to be optimal to derive x_1' to zero. Recalling (8) through (10),

the control is clearly $u^*(t) = 0$. It should be noted that in this case we have a <u>second</u> <u>order</u> <u>SVIC</u>. Considering (2), we must clearly also have $\phi(t) = 0$. However, since the choice of control is restricted by (10) and not (4), we may take $\eta_2(t) = 0$. The multiplier $\mu(t)$ is determined by the condition $\frac{\partial H}{\partial u} = 0$ and hence

$$\mu(t) = \frac{p_4(t)}{a_1 y} . \tag{85}$$

The condition $\mu(t) \geq 0$ yields that on a constrained subarc we must have

$$p_4(t) \geq 0, \tag{86}$$

which should be contrasted to the necessary condition for constrained subarc with $\phi = 0$ and $x_1 > 0$. It is also convenient to write (85) as

$$\mu(t) a_1 y = p_4(t). \tag{87}$$

Differentiating (87) with respect to time and considering (2) and (15), we obtain

$$\dot{\mu}(t) = \frac{1}{a_1}(a_1 p_1 - a_2 p_2). \tag{88}$$

It should at this point be noted that Gamkrelidze's condition $\dot{\mu} \leq 0$ does not apparently hold for higher order SVIC. A further differentiation (and use of (12) and (13)) yields

$$\mu(t) = \frac{p_3(t)}{a_1}(a_1b_1 - a_2b_2), \tag{89}$$

where we have used the fact that $\phi(t) = 0$.

We now will show that apparently due to this problem's special structure a necessary condition of optimality on a constrained subarc with $x_1(t) = 0$ is that

$$\mu(t) \geq 0. \tag{90}$$

We do this by considering the method of adjoining the state-variable constraint directly to the criterion functional with an additional Lagrange multiplier [2], [10]. When we add the state-variable constraint directly, the Hamiltonian is given by [10]

$$H(t,x_i,\hat{p}_i,u) = -\hat{p}_1\phi a_1 y - \hat{p}_2(1-\phi)\, a_2 y - \hat{p}_3(b_1x_1+b_2x_2) + \hat{p}_4 u + \hat{\mu}x_1, \tag{91}$$

where

$$\hat{\mu}(t) \begin{cases} = 0 & \text{for } x_1 > 0, \\ \geq 0 & \text{for } x_1 = 0, \end{cases}$$

and $\hat{p}_i$ and $\hat{\mu}$ denote the Lagrange multiplier when this method is used (as opposed to p_i and μ which denote the multipliers when Gamkrelidze's approach is used). The condition on the sign of $\hat{\mu}(t)$ when $x_1 = 0$ has been given by McIntyre and Paiewonsky [7] (see also [4]). (It is readily obtained using Valentine's method by considering the Legendre-Clebsch condition (see [11] for the corresponding development in mathematical programming).) The

adjoint system of differential equations for the dual variables is now

$$\frac{d\hat{p}_1}{dt} = b_1\hat{p}_3 - \hat{\mu}(t), \tag{92}$$

$$\frac{d\hat{p}_2}{dt} = b_2\hat{p}_3, \tag{93}$$

$$\frac{d\hat{p}_3}{dt} = \phi a_1\hat{p}_1 + (1-\phi)\ a_2\hat{p}_2, \tag{94}$$

$$\frac{d\hat{p}_4}{dt} = (a_1\hat{p}_1 - a_2\hat{p}_2)y. \tag{95}$$

On a constrained subarc on which $x_1(t) = 0$ (and $x_2 > 0$) for $t_1 \leq t \leq t_2$, we have

$$\frac{\partial H}{\partial u} = \frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0, \tag{96}$$

and the multiplier $\hat{\mu}(t)$ is determined by (96) and the condition

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right) = 0. \tag{97}$$

We find accordingly

$$\hat{\mu}(t) = \frac{\hat{p}_3(t)}{a_1}(a_1b_1 - a_2b_2). \tag{98}$$

The condition that $\hat{\mu}(t) \geq 0$ when $x_1 = 0$ for a finite interval of time yields that

$$a_1b_1 \geq a_2b_2, \tag{99}$$

since it is readily shown that $\hat{p}_3(t) > 0$. Thus, we have developed the same necessary condition for it to be optimal to drive x_1 to zero as for "inertialess" combat.

Let us now relate the above result for $\hat{\mu}$ to $\breve{\mu}$. Considering (89) and (98), we have

$$\frac{\breve{\mu}(t)}{p_3(t)} = \frac{\hat{\mu}(t)}{\hat{p}_3(t)}. \tag{100}$$

Assuming that $y(t) > 0$, consideration of the boundary condition for $p_3(t)$ and $\hat{p}_3(t)$ (18), the appropriate adjoint equation (and results for p_1, p_2, $\hat{p}_1$, and $\hat{p}_2$), and the corner conditions that

$$p_3(t_1-) = \hat{p}_3(t_1+), \tag{101}$$

and the condition [10]

$$\hat{p}_3(t_1-) = \hat{p}_3(t_1+), \tag{102}$$

we see that $p_3(t)$ and $\hat{p}_3(t)$ have the same sign. By (100) so do $\hat{\mu}(t)$ and $\breve{\mu}(t)$. Hence, we have shown that

$$\breve{\mu}(t) \geq 0 \quad \text{when} \quad x_1(t) = 0 \quad \text{for} \quad t_1 \leq t \leq t_2. \tag{103}$$

6. The Special Case of Unbounded Rates.

In our target selection problem (2), the rate of change of the fraction of Y fire directed at X_1 (i.e. $\frac{d\phi}{dt}$) is restricted.

Thus, we must have for $u = \frac{d\phi}{dt}$

$$- A \leq u \leq B. \tag{104}$$

Let us observe that for $- A = - \infty$ and $B = + \infty$ problem (2) reduces to problem (1), the "inertialess" combat problem. Accordingly, we would expect much to be similar in the solutions to these two problems. Indeed, this has been a result of our analysis: in both problems, for example, a necessary condition for it to be optimal to drive x_1 to zero is that $a_1 b_1 \geq a_2 b_2$.

It consequently is of interest to develop our previous results for (1) by considering (2) with $u = \frac{d\phi}{dt}$ unbounded. In this case, we must allow jumps the state variable ϕ. Such problems have only recently been studied in the literature [9], [12], [13]. We feel that the relationships between optimal target selection policies for "inertialess" and "inertial" combat should be further examined.

7. Discussion.

The results presented in this appendix are preliminary, being based on an initial cursory examination. We feel that this is an important and promising area and propose this to ONR as a future research task.

We saw that for both "inertialess" and "inertial" combat, we must necessarily have

$$a_1 b_1 \geq a_2 b_2,$$

in order for it to be optimal to drive x_1 to zero (while $x_2 > 0$ and before $t = T$). Furthermore, we also developed a necessary condition involving $v(t) = a_1p_1(t) - a_2p_2(t)$ for it to be optimal to have $\phi(t) = 0$ or 1 for a finite interval of time. Again, the results were similar to those for the "inertialess" combat problem. A significant result (see Section 4), however, was that for "inertial" combat an optimal policy for the distribution of fire over enemy target types is characterized by beginning to shift fire <u>before</u> target priorities (as measured by the sign of $v(t)$) change.

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Appendix K. Target Selection for Helmbold's General Attrition Structure.

1. Introduction.

One of the objectives of this research sponsored by ONR is to investigate the sensitivity of the optimal target selection policy to the form of the combat attrition model. We have initiated in Appendix I an investigation as to whether for the simplest target selection problem the optimal policy is sensitive to whether attrition is modelled as a deterministic or a stochastic process.

Let us now restrict our attention here to deterministic combat formulations. As discussed in Appendix H, various forms of Lanchester-type equations have been hypothesized to describe combat between two homogeneous forces. These formulations are readily extended to combat situations between heterogeneous forces. In Appendices A, B, C, E, F, G, and J we have considered target selection problems in which the rate of attrition of an enemy target type is directly proportional to the number of friendly firers. For this type of target-type attrition we saw that the optimal policy was always to concentrate all fire on a single target type. The index of the target type upon which all fire is concentrated may change (without the target type being annihilated) over the course of battle, but we always concentrate all our fire. In Appendix B we heuristically discussed why this is so in an intuitive fashion. In Appendices B and D we consider a target selection problem in which the rate of attrition of an enemy target

type is directly proportional to the product of the number of firers and the number of targets. For this type of target type attrition we saw that the optimal policy might be other than concentrating all fire on a single target type.

We were thus led to consider a general form of (deterministic) combat attrition in a simple target selection problem in order to study the sensitivity of the optimal policy. Specifically, we were interested in whether the optimal policy would be to always concentrate all fire on a single target type. In 1965 R. Helmbold proposed a modification of Lanchester's square law attrition model to incorporate inefficiencies of scale for the larger force when the force sizes are grossly unequal [3]. Although unfortunately not stated in this paper, we conjecture that this suggested modification was firmly based upon empirical evidence: Helmbold noted in [1] (see also [2]) that if one analyzed a large number of land battles as if each were combat between two homogeneous forces, then the ratio of attrition-rate coefficients was inversely proportional to the initial force ratio.

Hence, we decided to study the two-against-one target-type selection problem with combat attrition following Helmbold's general model. In this appendix we present preliminary results of our rather brief examination. We shall only develop the basic necessary conditions of optimality (and not address here the synthesis of optimal trajectories). These preliminary results indicate that when target-type

attrition follows Helmbold's general model, the optimal policy is always to concentrate all fire on a single target type. In other words, this attrition model apparently does not lead to a singular solution in the optimal control problem.

2. Helmbold's Modification of Lanchester's Equations.

In 1965 Helmbold [3] proposed a general Lanchester-type model in which the effectiveness of a force is dependent upon the force ratio. A special case (when each attrition rate is proportional to a product of terms one of which is a power of the force ratio) of Helmbold's general formulation is

$$\frac{dx}{dt} = -ax^c y^{1-c},$$

$$\frac{dy}{dt} = -by^c x^{1-c}, \tag{1}$$

with initial conditions

$$x(t=0) = x_0,$$

$$y(t=0) = y_0,$$

where a and b are Lanchester attrition-rate coefficients, $x(t)$ and $y(t)$ are force levels, and c is a parameter which takes on values between zero and one.

Helmbold gave the time solution to (1) as (for $c \neq 1$)

$$x(t) = \left\{x_0^{1-c} \cosh(1-c)\sqrt{ab}\, t - y_0^{1-c}\sqrt{\frac{a}{b}} \sinh(1-c)\sqrt{ab}\, t\right\}^{\frac{1}{1-c}},$$

and

$$y(t) = \{y_0^{1-c} \cosh(1-c)\sqrt{ab}\ t - x_0^{1-c}\sqrt{\frac{b}{a}} \sinh(1-c)\sqrt{ab}\ t\}^{\frac{1}{1-c}} \qquad (2)$$

He also stated that equations (1) also yield that state equation

$$b(x_0^{2-2c}-x^{2-2c}) = a(y_0^{2-2c}-y^{2-2c}), \qquad (3)$$

which yields for various values of the parameter c

$c = 0$	square law.
$c = 1/2$	linear law,
$c = 1$	"logarithmic" law.

When $c = 1$, (3) takes the form

$$b(\ell n\ x_0-\ell n\ x) = a(\ell n\ y_0-\ell n\ y). \qquad (4)$$

Helmbold also developed the "force-ratio" equation and obtained a solution.

Let us note that when $c = \frac{1}{2}$, equation (3) yields a "linear law." However, the reader should observe that although equation (3) becomes a "linear law" when $c = \frac{1}{2}$, equations (1) do not reduce to Lanchester's equation for area fire (see Appendix H), and consequently the time solution to (1) is not the same as that for the classical Lanchester area-fire equations (see [7] for the time solution).

3. The Optimal Control Problem.

Motivated by Helmbold's modification of Lanchester's equations, we consider the following optimal control problem:

$$\underset{\phi(t)}{\text{maximize}} \quad \{ry(T) - px_1(T) - qx_2(T)\} \quad \text{with} \quad T \quad \text{specified,}$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -\phi a_1 x_1{}^c y^{1-c},$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2{}^c y^{1-c},$$

$$\frac{dy}{dt} = -b_1 y^c x_1{}^{1-c} - b_2 y^c x_2{}^{1-c}, \tag{5}$$

$$x_1, x_2, y \geq 0 \quad \text{and} \quad 0 \leq \phi \leq 1,$$

where

$x_1(t), x_2(t), y(t)$ are force levels,

p, q, r are utilities assigned survivors,

a_1, a_2, b_1, b_2 are (constant) attrition-rate coefficients,

ϕ is the fraction of Y fire directed at X_1,

and c is a parameter which must satisfy $0 \leq c < 1$.

In our initial investigation here we will limit our discussion to the development of the basic necessary conditions of optimality.

4. Solution to Heterogeneous Combat Equations.

It is of interest to determine the state solution to

$$\frac{dx_1}{dt} = -\phi a_1 x_1^{c} y^{1-c},$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2^{c} y^{1-c},$$

$$\frac{dy}{dt} = -b_1 y^{c} x_1^{1-c} - b_2 y^{c} x_2^{1-c}, \qquad (6)$$

when ϕ = constant for $0 \le t \le t_1$. Equations (6) readily yield

$$z_0^{2} - z^{2}(t) = \{\phi a_1 b_1 + (1-\phi) a_2 b_2\}\{y_0^{2-2c} - (y(t))^{2-2c}\}, \qquad (7)$$

where

$$z(t) = b_1 x_1^{1-c} + b_2 x_2^{1-c},$$

and

$$z(t=0) = z_0.$$

As we have seen in Appendices A, F, and G, the state solution (7) is useful in synthesizing optimal trajectories. A similar simple solution also arises in the case of n-versus-one combat according to Helmbold's general attrition model.

5. Development of Basic Necessary Conditions of Optimality.

We will focus on the development of necessary conditions of optimality. It is convenient to consider state variable inequality

constraints equivalent to $x_1 \geq 0$, $x_2 \geq 0$ for (5). Instead of $x_1 \geq 0$, we consider

$$- x_1^{1-c} \leq 0,$$

and similarly

$$- x_2^{1-c} \leq 0,$$

where $c \in [0,1)$ so that $1 - c > 0$. When $x_1 = 0$, we must therefore have

$$\frac{d}{dt}(-x_1^{1-c}) \leq 0, \quad \text{so that}$$

$$(1-c)\phi a_1 y^{1-c} \leq 0 \quad \text{when} \quad x_1 = 0, \tag{8}$$

and similarly

$$(1-c)(1-\phi)a_2 y^{1-c} \leq 0 \quad \text{when} \quad x_2 = 0. \tag{9}$$

Following Gamkrelidze (see Chapter VI of [6])(see also section 4.b.(3) of Appendix E), the Hamiltonian is given by

$$H(t,x_i,p_i,\phi) = - p_1\phi a_1 y^{1-c}x_1{}^c - p_2(1-\phi)a_2 y^{1-c}x_2{}^c$$

$$- p_3 y^c(b_1 x_1{}^{1-c}+b_2 x_2{}^{1-c}) - \mu_1(t)\phi a_1 y^{1-c} - \mu_2(t)(1-\phi)a_2 y^{1-c}, \tag{10}$$

where

$$\mu_i(t) \begin{cases} = 0 & \text{for } x_i > 0, \\ \geq 0 & \text{for } x_i = 0, \end{cases}$$

and p_1 is the dual variable corresponding to x_1. The adjoint system of differential equations for the dual variables is

$$\frac{dp_1}{dt} = c\phi a_1 p_1 x_1^{c-1} y^{1-c} + (1-c) b_1 p_3 x_1^{-c} y^c, \tag{11}$$

$$\frac{dp_2}{dt} = c(1-\phi) a_2 p_2 x_2^{c-1} y^{1-c} + (1-c) b_2 p_3 x_2^{-c} y_c, \tag{12}$$

$$\frac{dp_3}{dt} = (1-c) y^{-c} \{\phi a_1 (p_1 x_1^{c} + \mu_1) + (1-\phi) a_2 (p_2 x_2^{c} + \mu_2)\}$$

$$+ c p_3 y^{c-1} (b_1 x_1^{1-c} + b_2 x_2^{1-c}). \tag{13}$$

When $x_1, x_2 > 0$, the control low is determined by the maximum principle. Hence, we consider

$$\underset{0 \le \phi \le 1}{\text{maximize}}\ H(t, x_i, p_i, \phi).$$

This yields

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad a_1(-p_1(t)) x_1^{c} > a_2(-p_2(t)) x_2^{c}, \\ 0 & \text{for} \quad a_1(-p_1(t)) x_1^{c} < a_2(-p_2(t)) x_2^{c}. \end{cases} \tag{14}$$

We must further investigate the possibility of a singluar subarc [4], [5] on which $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time (see discussion in Appendix D).

a. Investigation of Singular Subarcs.

As is well-known, since the Hamiltonian (10) is a linear function of the control variable ϕ, the maximum principle does not determine the control when the coefficient of ϕ in H vanishes for a finite interval of time [4], [5]. On such a singular subarc we must have

$$\frac{\partial H}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0. \qquad (15)$$

The condition $\frac{\partial H}{\partial \phi} = 0$ yields that on a singluar subarc we must have

$$a_1 p_1 x_1^{\ c} = a_2 p_2 x_2^{\ c}, \qquad (16)$$

where we have observed that $y > 0$. Combining (16) with the expression for $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right)$ we obtain

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = (1-c)yp_3(a_2 b_2 - a_1 b_1) \quad \text{when} \quad \frac{\partial H}{\partial \phi} = 0. \qquad (17)$$

Assuming that $c \neq 1$ and $a_1 b_1 \neq a_2 b_2$, we see that $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ when $\frac{\partial H}{\partial \phi} = 0$ implies that we must have $p_3(t) = 0$. While we have obtained no contradiction that $p_3(t) = 0$ for a finite interval of time on a singular subarc, considering the interpretation of the dual variable $p_3 = \frac{\partial S}{\partial y}$ where S is the optimal value function, it intuitively seems impossible that $p_3(t) = 0$ for a finite interval of time. (This would imply that there is no marginal return for increasing the Y-force level.) Thus, we tentatively conclude that there is no singular solution, and the optimal policy is $\phi^* = 0$ or 1 (with possible changes over time). Hence, we see that for Helmbold's

general attrition model the optimal tactic is apparently always to concentrate all fire on a single target type.

b. Optimality Conditions for Force Level to be Driven to Zero.

Let us now investigate whether it is a good policy to annihilate a force type in this model. (The reader can find a discussion of the theory of state variable inequality constraints that we apply here in Appendix E.) Without loss of generality, we may consider a constrained subarc on which $x_1(t) = 0$ (and $x_3 > 0$) for $t_1 \le t \le T$. The control is clearly $\phi^*(t) = 0$. The multiplier $\mu_1(t)$ is determined by the condition $\frac{\partial H}{\partial \phi} = 0$ and hence

$$\mu_1(t) = \frac{1}{a_1}(-p_1 a_1 x_1{}^c + p_2 a_2 x_2{}^c). \qquad (18)$$

Differentiating (19) and combining with (5), (11), and (12), we find that

$$\dot{\mu}_1(t) = -\frac{(1-c)}{a_1} p_3 y^c (a_1 b_1 - a_2 b_2), \qquad (19)$$

so that Gamkrelidze's condition $\dot{\mu}_1(t) \le 0$ [6] is only satisfied on a constrained subarc with $x_1 = 0$ when

$$a_1 b_1 \ge a_2 b_2 \qquad (20)$$

since we would expect to have $p_3(t) \ge 0$.

However, even if (20) is satisfied, it is never an optimal policy to drive x_1 to zero (unlike the corresponding situation for the Isbell-Marlow problem discussed in Section 4.b.(1) of Appendix E).

The proof of this statement is by contradiction. The necessary condition that $\mu_1(t) \geq 0$ where μ_1 is given by (18) combined with $x_1 = 0$ yields that

$$p_2(t) \geq 0 \text{ for } \quad t_1 \leq t \leq T. \tag{21}$$

However, for $x_2(T) > 0$ we have the boundary condition

$$p_2(t{=}T) = -q. \tag{22}$$

By the inconsistency of (21) and (22)(both of which are necessary conditions of optimality) it can never be an optimal policy to drive x_1 to zero (and have $x_2 > 0$).

6. Discussion.

The results presented in this appendix are preliminary, being based on an initial cursory examination. We feel that this is an important and promising area and propose this to ONR as a future research task.

We tentatively concluded that the optimal policy was always to concentrate all fire on one target type (the same optimal policy as for the one-versus-n problem (see Appendix C) with a "square-law attrition" of target types). In other words, our analysis so far has disclosed no singular solution [5] to (5). Considering the time solution given by (2) to the combat equations (1), we see (heuristically) that the time solution to equations (1) when $c = \frac{1}{2}$ is closer in form to that for a "square-law" battle than that for Lanchester's classical

area-fire equations [7]. Hence, considering the results of Appendix D, our tentative conclusion here on concentration of fire as an optimal policy is intuitively appealing, although we feel that more work remains to be done.

We finally saw that if $x_2 > 0$, then it was not an optimal policy to drive x_1 to zero. Considering (14), this makes sense, since as x_1 goes to zero we eventually must have $a_1(-p_1(t))x_1^c < a_2(-p_2(t))x_2^c$. Intuitively this policy also makes sense, since the motivation behind Helmbold's formulation (1) is inefficiencies of scale for the larger force. Hence, as x_1 "becomes small enough" the Y-force is more efficient at destroying X_2.

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The mathematical theory of optimal control/differential games is used to study the structure of optimal allocation policies for some tactical allocation problems with combat described by Lanchester-type equations of warfare. Both deterministic and stochastic attrition processes are considered. For the optimal control of deterministic Lanchester-type attrition processes, a general solution algorithm for the synthesis of the optimal policy is developed. Optimal allocation policies are developed for numerous one-sided optimization problems of tactical interest in order to study the dependence of the structure of these optimal policies on model form. Consideration has been given to singular extremals, multiple extremals (including dispersal surfaces), and state variable inequality constraints. It is shown how to apply the theory of state variable inequality constraints to determine the optimal control of deterministic Lanchester-type processes in order to treat non-negativity restrictions on force levels and thus to study the dependence of optimal policies upon the force levels. Various attrition models are considered (reflecting different assumptions as to target acquisition process, command and control capabilities, target engagement process, variations in range capabilities of weapon systems). Solutions are developed for Lanchester-type equations of modern warfare with variable attrition-rate coefficients. The optimal control of the Lanchester stochastic process is studied.

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